J. Fluid Mech. (1982), *vol.* 116, *pp.* 1–29
Printed in Great Britain

Theory of the pressure–strain rate. Part 2. Diagonal elements

By J. WEINSTOCK

National Oceanic and Atmospheric Administration, Aeronomy Laboratory, Boulder, Colorado 80303

(Received 17 February 1981 and in revised form 23 June 1981)

A theoretical calculation is made of (the diagonal elements of) pressure–strain-rate calculation $\rho_0^{-1}\langle p[\nabla \mathbf{u} + (\nabla \mathbf{u})^T]\rangle$ for a simple turbulent shear flow. This calculation parallels a previous calculation of the off-diagonal element. The calculation is described as follows. (1) Beginning with the Navier–Stokes equation, an expression for the (diagonal) pressure–strain-rate term is derived analytically in terms of measurable quantities (velocity spectra) – this derivation makes use of a cumulant discard. (2) It is proved that, to lowest order in the spectral anisotropy, the diagonal pressure–strain-rate term is linearly proportional to the diagonal Reynolds-stress elements. (3) A formula is derived for the proportionality constants (Rotta constants) in terms of arbitrary spectra. (4) This formula is used to calculate theoretically the numerical value of Rotta's constant C_{ii} for models of velocity spectra (the variation of C_{ii} with variations of spectral shapes and of Reynolds number are also determined). (5) Deficiencies and limitations of Rotta's model are identified and discussed.

It is found that Rotta's expression for $2\rho_0^{-1}\langle p\, \partial u_i/\partial i\rangle$ is only valid for special spectra. Surprisingly large deviations of Rotta's expression from theory are found for a more complex spectra thought to be typical of simple shear flow. In addition, it is found that C_{xz} is intrinsically and quantitatively different from C_{ii} because the latter depends importantly on the large-wavenumber part of the spectrum (the inertial subrange) whereas the former does not. The numerical ratio C_{zz}/C_{xz} is calculated theoretically and shown to be about 2 for the zero-moment model. It is concluded that a linear term in the stress anisotropy as proposed by Rotta does not always exist. The deviation of Rotta's model from theory is understood by distinguishing between the spectral anisotropy and the stress anisotropy.

For the zero-moment spectral model, where the Rotta relation is valid, it is found that C_{ii} varies significantly with large Reynolds number but is rather insensitive to the large-wavelength part of the spectrum.

1. Introduction

In a previous paper (Weinstock 1980), hereinafter referred to as I, an off-diagonal element, $2A_{xz}$, of the pressure–strain-rate tensor was theoretically calculated directly from the Navier–Stokes equation. The theoretical $2A_{xz}^N$ was then compared with Rotta's model (1951) and with empirical determinations of the pressure–strain rate. The purpose of the present paper is to calculate $2A_{ii}^N$, the diagonal elements of the pressure–strain-rate tensor. The goals of this calculation parallel those in I. These are: (1) to derive analytically an expression for the diagonal elements of the pressure–strain rate in terms of measurable quantities (the velocity spectra); (2) to prove that (to lowest order in the spectral anisotropy) the pressure–strain rate is linearly proportional to the Reynolds stress; (3) to derive a formula for the constants of proportionality (Rotta constants) in terms of arbitrary velocity spectra; (4) to use this formula to calculate analytically Rotta's constant for models of energy spectra in nearly homogeneous shear flows and investigate the variations of these constants caused by variations of the spectra and flow parameters; and (5) to assess the validity or limitations of Rotta's model by comparison with the theory.

In I, it was not possible to investigate or assess Rotta's model to any extent because only an off-diagonal element of the pressure–strain rate was calculated. A more extensive assessment is possible here because the three diagonal elements are calculated as well. Comparisons can also be made with empirical and experimental determinations of the pressure–strain rate (e.g. Reynolds 1976; Hanjalic & Launder 1972; Lumley & Khajeh-Nouri 1974; Launder, Reece & Rodi 1975; Lumley & Newman 1977; Comte-Bellot & Corrsin 1966; Champagne, Harris & Corrsin 1970; Harris, Graham & Corrsin 1977; and others).

1.1. *Plan and assumptions of the calculation*

The method of our calculation from the Navier–Stokes equation is the same as described in §1.1 of I. It is outlined as follows. Nonlinear expressions for the velocity fluctuations $\mathbf{u}$ and pressure fluctuations p are derived from a straightforward solution of the Navier–Stokes equation. These expressions for $\mathbf{u}$ and p allow us to relate the pressure–strain-rate tensor $\rho_0^{-1}(\langle p\nabla\mathbf{u}\rangle + \langle p\nabla\mathbf{u}\rangle^T)$ (ρ_0 is the density and the superscript T denotes the transpose) to a two-point fourth-order velocity correlation. This correlation is then evaluated analytically in terms of the single-point velocity covariance (i.e. the Reynolds stress) by a cumulant discard. The distinctions between this calculation and those based on the direct-interaction approximation (Herring 1974; Leslie 1973; and Schumann & Herring 1976) were mentioned in I. Briefly, we are less ambitious than these authors because we do not calculate the energy spectra as they do, but, instead, simply derive the pressure–strain rate in terms of the spectra. Our calculation is less self-contained, but encompasses a wider range of turbulence states and is entirely analytic.

The simplifying assumptions of the calculation are the same as in I. Our intention is to consider a simplified shear flow so that the underlying approximations will be masked as little as possible by the complexity of that flow. We thus restrict ourselves to: (1) a uni-directional mean flow $\mathbf{U} = (U_0(z), 0, 0)$ in a Cartesian co-ordinate system (x, y, z); (2) $\partial\mathbf{U}/\partial z$ and all ensemble-average quantities (correlation functions) are assumed to vary only a little in space and time on scales $2\pi k_L^{-1}$ and τ_L, respectively, where k_L is the characteristic wavenumber of the energy-containing region of the spectrum and τ_L is the Lagrangian integral time scale; and (3) large Reynolds number. The calculation can be readily generalized to more complex flow geometrics if that should prove desirable. A correction for low-Reynolds-number flow is given in I (appendix D).

The organization of this paper is as follows: In §2 the pressure–strain rate $2A_{ii}^N$ is derived in terms of the velocity spectrum $\mathbf{S}(\mathbf{k})$ (a measurable quantity) to general order in anisotropy. In §3, $2A_{ii}^N$ is expressed explicitly in terms of the Reynolds stress $\langle \mathbf{uu} \rangle \equiv \int d\mathbf{k}\mathbf{S}(\mathbf{k})$ to first order in the spectral anisotropy. Theoretical derivations of Rotta's expression for $2A_{ii}^N$ and of the numerical value of Rotta's constant C_{ii} are given in §§4 and 4.1. There it is shown that Rotta's expression for $2A_{ii}$ is only valid for a special class of spectra including what is referred to as the zero-moment model. A fundamental difference between $2A_{zz}^N$ and $2A_{xz}^N$, and between C_{zz} and C_{xz}, is pointed out in §4.2, and an expression is derived for the ratio C_{xz}/C_{zz}. Large deviations from Rotta's expression are found in §4.3 for a more general spectral model in which the maximum (spectral peak) of S_{xx} occurs at a wavenumber different from that of the maximum of S_{zz}. This spectrum, referred to as the higher-moment model, was found to occur in a nearly homogeneous shear flow by Kaimal *et al.* (1972). A discussion of the theoretical deficiencies and limitations of Rotta's model is given in §5. The deviations of Rotta's model from theory is understood by distinguishing between the spectral anisotropy and the stress anisotropy. A selective comparison with experiments is made in §6, a discussion of errors in the theory is given in §7, and a summary with conclusions is given in §8.

2. Derivation of pressure–strain rate in terms of velocity spectra

In this section the diagonal elements of the pressure–strain-rate tensor are derived in terms of measurable velocity spectra. The pressure–strain rate tensor occurs in the Reynolds-stress transport equation and is defined by

$$\rho_0^{-1}\langle p[\nabla\mathbf{u} + (\nabla\mathbf{u})^T]\rangle,$$

where $\mathbf{u} \equiv \mathbf{u}(\mathbf{r}, t)$ is the fluctuating part of the fluid velocity at position $\mathbf{r}$ at time t, $p = p\ (\mathbf{r}, t)$ is the fluctuating part of the pressure at $\mathbf{r}$ and t, and the angle brackets denote an ensemble average (mean value). Our object is to calculate the diagonal elements

$$\frac{2}{\rho_0}\left\langle p\frac{\partial u_x}{\partial x}\right\rangle,\quad \frac{2}{\rho_0}\left\langle p\frac{\partial u_y}{\partial y}\right\rangle,\quad \frac{2}{\rho_0}\left\langle p\frac{\partial u_z}{\partial z}\right\rangle,$$

where x, y, z are the three Cartesian co-ordinates, and u_i is the component of $\mathbf{u}$ along i. First we calculate $2\rho_0^{-1}\langle \partial u_z/\partial z\rangle$; afterwards, it will be simple to obtain the other diagonal elements (given in appendix B). Much of our calculation is very similar to the previous calculation of the off-diagonal element $2\rho_0^{-1}\langle p\,\partial u_x/\partial z\rangle$ given in I, and, for the sake of brevity, some proofs of our derivation will merely be quoted from there. However, sufficient details will be included to make the present derivation complete and coherent.

It is most convenient to evaluate $\langle p\,\partial u_z/\partial z\rangle$ in terms of its Fourier transform expressed as follows:

$$\langle p\,\partial u_z/\partial z\rangle = -\frac{1}{(2\pi)^3 V}\int d\mathbf{k}\langle u_z^*(\mathbf{k}, t)\,ik_z p(\mathbf{k}, t)\rangle, \tag{1}$$

where the asterisk * denotes the complex conjugate, V is the volume of the system, and $u_z(\mathbf{k})$ and $p(\mathbf{k})$ denote the Fourier transforms of u_z and p defined by

$$u_z(\mathbf{k}, t) \equiv \int d\mathbf{r} u_z(\mathbf{r}, t)\exp i\mathbf{k}.\mathbf{r}$$

and

$$p(\mathbf{k}, t) \equiv \int d\mathbf{r} p(\mathbf{r}, t)\exp i\mathbf{k}.\mathbf{r}.$$

Both p and u_z are obtained from the Navier–Stokes equation. The fluctuating part of that equation is given by

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}+\mathbf{U}).\nabla\mathbf{u} = \langle \mathbf{u}.\nabla\mathbf{u}\rangle - \frac{\nabla p}{\rho_0} + \mathbf{u}.\nabla\mathbf{U} + \nu\nabla^2\mathbf{u}, \tag{2}$$

where $\mathbf{U}$ is the mean flow velocity, ν is the molecular viscosity, and ρ_0 is the fluid density (assumed to be constant). Equation (2) is obtained from the Navier–Stokes equation by subtracting out its average.

First we obtain p, and afterwards $\mathbf{u}$. A useful expression for p is obtained, as is well known, by taking the divergence of (2) and using $\nabla.\mathbf{u} = 0$ (incompressibility):

$$\frac{\nabla^2 p(t)}{\rho_0} = -\nabla.(\mathbf{u}.\nabla\mathbf{u})' - 2\frac{\partial u_z}{\partial x}\frac{\partial U_0}{\partial z}, \tag{3}$$

where

$$(\mathbf{u}.\nabla\mathbf{u})' \equiv \mathbf{u}.\nabla\mathbf{u} - \langle \mathbf{u}.\nabla\mathbf{u}\rangle \tag{4}$$

is the fluctuating part of $\mathbf{u}.\nabla\mathbf{u}$, and we have used the idealized flow $\mathbf{U} = [U_0(z), 0, 0]$ so that $\nabla.(\mathbf{U}.\nabla\mathbf{u}+\mathbf{u}.\nabla\mathbf{U}) = 2(\partial u_z/\partial x)(\partial U_0/\partial z)$. An expression for p is now obtained by taking the Fourier transform of both sides of (3) and neglecting the spatial variation of $\partial U_0/\partial z$ compared with that of u_z. The result is

$$\rho_0^{-1} p(\mathbf{k}, t) = N(\mathbf{k}, t) + \frac{2ik_x}{k^2} u_z(\mathbf{k}, t)\frac{\partial U_0}{\partial z}, \tag{5}$$

where $\mathbf{u}(\mathbf{k}, t)$ is the Fourier transform of $\mathbf{u}$, and $N(\mathbf{k}, t)$ is the transform of the non-linear fluctuation term $\nabla.(\mathbf{u}.\nabla\mathbf{u})'$ given explicitly by

$$N(\mathbf{k}, t) \equiv -\int \frac{d\mathbf{k}_1}{(2\pi)^3}\frac{\mathbf{k}}{k}.[\mathbf{u}(\mathbf{k}_1, t)\,\mathbf{u}(\mathbf{k}-\mathbf{k}_1, t) - \langle \mathbf{u}(\mathbf{k}_1, t)\,\mathbf{u}(\mathbf{k}-\mathbf{k}_1, t)\rangle].\frac{\mathbf{k}}{k}. \tag{6}$$

In (6), we have used the inverse Fourier transform

$$\mathbf{u}(\mathbf{r}, t) = (2\pi)^{-3}\int d\mathbf{k}_1 \mathbf{u}(\mathbf{k}_1, t) \exp i\mathbf{k}.\mathbf{r}. \tag{7}$$

Substitution of (5) in (1) yields the pressure–velocity correlation in the familiar form

$$\left.\begin{aligned} 2\rho_0^{-1}\langle p(t)\,\partial u_z(t)/\partial z\rangle &= 2A_{zz}^N + 2A_{zz}^M, \\ A_{zz}^N &\equiv -\frac{i}{(2\pi)^3 V}\int d\mathbf{k} k_z \langle u_z^*(\mathbf{k}, t)\,N(\mathbf{k}, t)\rangle, \\ A_{zz}^M &\equiv \frac{2}{(2\pi)^3 V}\int d\mathbf{k}\left(\frac{k_x k_z}{k^2}\right)\langle u_z^*(\mathbf{k}, t)\,u_z(\mathbf{k}, t)\rangle \frac{\partial U_0}{\partial z}, \end{aligned}\right\} \tag{8}$$

where $2A_{zz}^N$ is seen to be the contribution to the pressure–strain rate coming from the turbulent-velocity fluctuation part of p, and $2A_{zz}^M$ is the contribution from the mean-velocity (mean-strain) part of p. Note that the expression for A_{zz}^N differs from the analogous expression for A_{xz}^N given in I only because k_z occurs in the integral instead of k_x.

It is the term A_{zz}^N in (8) for which Rotta (1951) proposed his model. This term contains the third-order (triple-point) velocity correlation $\langle u_z^* N\rangle$ and, hence, presents a familiar problem of closure. A (closure) calculation of $\langle u_z^* N\rangle$ is given in I in great detail. In that calculation, u_z is expressed as a second-order velocity fluctuation (which is obtained by formally solving the Navier–Stokes equation), so that $\langle u_z^* N\rangle$ can be expressed as a fourth-order velocity correlation. A cumulant-discard approximation is then applied directly to that fourth-order correlation to obtain $\langle u_z^* N\rangle$ in terms of a two-point covariance (closure). Since the details of that calculation are already given in I, we shall present only the result here. Thus, the expression for A_{zz}^N is obtained from equation (27) of I by multiplying the integrand of (27) by k_z/k_x (the fact that A_{zz}^N and A_{xz}^N are related by the factor k_z/k_x in their **k**-integrals is seen by comparing the present equation (8) with equation (11) of I). We thus have immediately

$$A_{zz}^N = -2\int \frac{d\mathbf{k}_1}{(2\pi)^3}\int \frac{d\mathbf{k}}{(2\pi)^3}\frac{\tau_c \mathbf{b}^{zz}(\mathbf{k})}{(1+\delta)} : \mathbf{S}(\mathbf{k}_2)\,\mathbf{S}(\mathbf{k}_1) : \frac{\mathbf{k}^2}{k^2}, \tag{9}$$

where

$$\mathbf{S}(\mathbf{k}) \equiv \langle \mathbf{u}(\mathbf{k},t)\,\mathbf{u}^*(\mathbf{k},t)\rangle\, V^{-1}, \quad \int \frac{d\mathbf{k}}{(2\pi)^3}\mathbf{S}(\mathbf{k}) = \langle \mathbf{uu}\rangle, \tag{10}$$

(a tensor) is the velocity covariance spectrum at wavenumber **k** and time t, $\mathbf{k}_2 \equiv \mathbf{k}-\mathbf{k}_1$, and $\mathbf{b}^{zz}(\mathbf{k})$, τ_c, and δ are defined by

$$\left.\begin{aligned} \mathbf{b}^{zz}(\mathbf{k}) &\equiv k_z \mathbf{k}\hat{\mathbf{z}} - k_z^2 \mathbf{k}^2/k^2, \\ \tau_c &\equiv (\tfrac{1}{2}\pi)^{\frac{1}{2}}[(\mathbf{k}_1^2+\mathbf{k}_2^2) : \langle \mathbf{uu}\rangle]^{-\frac{1}{2}}, \\ \delta &\equiv -(kv_0)^{-1}\frac{k_x k_z}{k^2}\frac{\partial U_0}{\partial z}. \end{aligned}\right\} \tag{11}$$

Here $v_0^2 \equiv \frac{1}{3}\langle \mathbf{u}\,.\,\mathbf{u}\rangle$ is the mean-square fluctuating velocity of the turbulence, and τ_c is recognized to be characteristic of the decay time of a spectrum at wavenumber $(k_1^2+k_2^2)^{\frac{1}{2}}$ (e.g. Kraichnan 1959). (Note that (9) differs from the expression for A_{xz}^N given in (27) of I in that $\mathbf{b}^{zz}(\mathbf{k}) \equiv (k_z/k_x)\,\mathbf{b}$ occurs instead of $\mathbf{b}$. Note, too, that $(2\pi)^{-3}\int d\mathbf{k}\mathbf{S}(\mathbf{k})$ equals $\langle \mathbf{uu}\rangle$ because the spectrum $\mathbf{S}(\mathbf{k})$ is defined as the Fourier transform of the two-point velocity covariance.)

Equation (9) determines the pressure–velocity correlation A_{zz}^N in terms of a measurable quantity (the velocity spectrum $\mathbf{S}$). This equation is a principal relationship of our work. No approximations have been made about the anisotropy, so that (9) is valid to all orders in the anisotropy. The main limitation of (9) is to slow variation of mean quantities in space and time, and the main approximation is the cumulant discard discussed in I (see appendix A of I).

If the velocity spectrum were known by theory or experiment, it would then be straightforward to evaluate (9) and thereby determine the pressure–strain rate (to general order in the anisotropy). As pointed out in I, some aspects of $\mathbf{S}$ are known fairly well and other aspects of $\mathbf{S}$ can be modelled to permit a useful evaluation of (9). The dependence of A_{zz}^N on variations of the models, and on the flow parameters, can then be tested. In the following sections we make such an evaluation of (9) to first order in anisotropy to determine A_{zz}^N in terms of the Reynolds stress. This theoretical A_{zz}^N is afterwards compared with Rotta's model and with experiment.

Let us next continue to the evaluation of (9) to first order in the anisotropy.

3. A_{zz}^N to first order in anisotrophy and stress

The purpose of this section is to evaluate A_{zz}^N, as given by (9), explicitly in terms of the Reynolds stress. This evaluation is made to first order in the anisotropy, and the resulting expression is compared with Rotta's model. Our calculation of A_{zz}^N parallels, and is very similar to, the calculation of A_{xz}^N already given in I. The difference, as mentioned above, arises from the factor k_z/k_x.

To expand A_{zz}^N in powers of anisotropy, we divide $\mathbf{S}(\mathbf{k})$ in (9) into an isotropic part $\mathbf{S}(\mathbf{k})^I$ and an anisotropic deviation $\mathbf{S}(\mathbf{k})^A$ as was done in I:

$$\left.\begin{aligned} \mathbf{S}(\mathbf{k}) &\equiv \mathbf{S}(\mathbf{k})^I + \mathbf{S}(\mathbf{k})^A, \\ \mathbf{S}(\mathbf{k})^I &\equiv 2\pi^2\left(\mathbf{I} - \frac{\mathbf{kk}}{k^2}\right)\frac{E(k)}{k^2}. \end{aligned}\right\} \tag{12}$$

$\mathbf{I}$ is the identity matrix, and $E(k)$ is the scalar energy spectrum, which satisfies

$$\int dk E(k) = \tfrac{1}{2}\langle \mathbf{u}.\mathbf{u}\rangle \equiv \tfrac{3}{2}v_0^2. \tag{12$'$}$$

Similarly, the stress tensor is divided into an isotropic part $v_0^2\mathbf{I}$ and an anisotropic part $\mathbf{a}$:

$$\langle \mathbf{uu}\rangle = v_0^2\mathbf{I} + \mathbf{a}. \tag{13}$$

We emphasize that the spectral anisotropy $\mathbf{S}^A$ is more general than the stress anisotropy $\mathbf{a}$. This distinction, which is illustrated by the fact that a zero value of $\mathbf{a}$ does not imply a zero value of $\mathbf{S}^A$, is found in §4.3 to cause a departure from Rotta's model. Note, too, that the definition of $\mathbf{S}^I(\mathbf{k})$ in (12) and (12$'$) implies that

$$\operatorname{tr}(2\pi)^{-3}\int d\mathbf{k}\,\mathbf{S}(\mathbf{k}) = \operatorname{tr}(2\pi)^{-3}\int d\mathbf{k}\,\mathbf{S}^I(\mathbf{k}) = \langle \mathbf{u}.\mathbf{u}\rangle,$$

which means that there is no net energy in $\operatorname{tr}\mathbf{S}^A$; i.e. the anisotropy corresponds to more energy in one direction than another, not to a change in the total energy.

Equation (9) can now be linearized by substituting (12) and (13) and neglecting all second- and higher-order terms in the anisotropy ($\mathbf{S}^A$ and $\mathbf{a}$). This linearization yields

$$A_{zz}^N = -2\int\frac{d\mathbf{k}}{(2\pi)^3}\int\frac{d\mathbf{k}_1}{(2\pi)^3}(\tfrac{1}{2}\pi)^{\frac{1}{2}}\left\{\frac{\mathbf{b}^{zz}:[\mathbf{S}(\mathbf{k}_1)\,\mathbf{S}(\mathbf{k}_2)^I + \mathbf{S}(\mathbf{k}_1)^I\,\mathbf{S}(\mathbf{k}_2)]:\mathbf{k}^2}{(k_1^2+k_2^2)^{\frac{1}{2}}\,k^2v_0} - \frac{\mathbf{b}^{zz}:\mathbf{S}(\mathbf{k}_1)^I\mathbf{S}(\mathbf{k}_2)^I:\mathbf{k}^2}{2(k_1^2+k_2^2)^{\frac{3}{2}}\,v_0^3k^2}[(\mathbf{k}_1^2+\mathbf{k}_2^2):\mathbf{a}]\right\}, \tag{14}$$

which gives A_{zz}^N to first order in the spectral anisotropy (the quantity δ is nonlinear and very small).

It is not difficult to express the right-hand side of (14) in terms of the Reynolds stress. A useful simplification for this purpose comes from incompressibility:

$$\mathbf{k}_2.\mathbf{S}(\mathbf{k}_2) = 0, \quad \mathbf{k}.\mathbf{S}(\mathbf{k}_2) = (\mathbf{k}_1+\mathbf{k}_2).\mathbf{S}(\mathbf{k}_2) = \mathbf{k}_1.\mathbf{S}(\mathbf{k}_2).$$

The first term in the integrand of (14) can thus be expressed, with (12), as

$$\mathbf{b}^{zz}:\mathbf{S}(\mathbf{k}_1)\,\mathbf{S}(\mathbf{k}_2)^I:\mathbf{k}^2k^{-2} = \left[k_z\mathbf{k}_2.\mathbf{S}(\mathbf{k}_1).\hat{\mathbf{z}} - \frac{k_z^2}{k^2}(\mathbf{k}_2.\mathbf{S}(\mathbf{k}_1).\mathbf{k}_2)\right] \times\left[k_1^2 - \frac{(\mathbf{k}_1.\mathbf{k}_2)^2}{k_2^2}\right]\frac{2\pi^2E(k_2)}{k^2k_2^2}, \tag{15}$$

and the second term is expressed, after interchanging the dummy variables of integration ($\mathbf{k}_1 \to \mathbf{k}_2$, $\mathbf{k}_2 \to \mathbf{k}_1$), as

$$\mathbf{b}^{zz} : \mathbf{S}(\mathbf{k}_2)^I \mathbf{S}(\mathbf{k}_1) : \mathbf{k}^2 k^{-2} = \left[k_z k_{1z} - \frac{k_z k_{2z} \mathbf{k}_1 . \mathbf{k}_2}{k_2^2} - \frac{k_z^2}{k^2} \left(k_1^2 - \frac{(\mathbf{k}_1 . \mathbf{k}_2)^2}{k_2^2} \right) \right] \times [\mathbf{k}_2 . \mathbf{S}(\mathbf{k}_1) . \mathbf{k}_2] \frac{2\pi^2 E(k_2)}{k^2 k_2^2}. \quad (16)$$

The third term in (14) has been evaluated in a straightforward, though lengthy, integration, and we have found it very small in comparison with the first two terms; it is henceforth neglected.

To express (15) and (16) in terms of stress-tensor elements $\langle u_i u_j \rangle$, let us expand out the spectral terms $k_z \mathbf{k}_2 . \mathbf{S}(\mathbf{k}_1) . \hat{\mathbf{z}}$ and $\mathbf{k}_2 . \mathbf{S}(\mathbf{k}_1) . \mathbf{k}_2$ in (15) and (16) as follows:

$$k_z \mathbf{k}_2 . \mathbf{S}(\mathbf{k}_1) . \hat{\mathbf{z}} = k_z (k_{2x} S_{xz} + k_{2y} S_{yz} + k_{2z} S_{zz}), \quad (17)$$

$$\mathbf{k}_2 . \mathbf{S}(\mathbf{k}_1) . \mathbf{k}_2 = k_{2x}^2 S_{xx} + k_{2y}^2 S_{yy} + k_{2z}^2 S_{zz} + 2k_{2x} k_{2y} S_{xy} + 2k_{2x} k_{2z} S_{xz} + 2k_{2y} k_{2z} S_{yz}, \quad (18)$$

where to condense notation we use S_{ij} to denote $S_{ij}(\mathbf{k}_1)$ $(i, j = x, y, z)$ so that, for example, $S_{xx} \equiv S_{xx}(\mathbf{k}_1)$. It is desirable to eliminate the off-diagonal spectral elements S_{xy}, S_{xz}, S_{yz} because Rotta's hypothesis predicts that A_{zz}^N should depend on only the diagonal elements S_{ii}, in the form $\langle u_i u_i \rangle \equiv (2\pi)^{-3} \int d\mathbf{k}_1 S_{ii}(\mathbf{k}_1)$. Expressions for the off-diagonal elements (in terms of the diagonal elements) are easily obtained from the incompressibility conditions

$$\mathbf{k}_1 . \mathbf{S}(\mathbf{k}_1) . \hat{\mathbf{x}} \equiv k_{1x} S_{xx} + k_{1y} S_{yx} + k_{1z} S_{zx} = 0, \quad \mathbf{k}_1 . \mathbf{S}(\mathbf{k}_1) . \hat{\mathbf{y}} = 0, \quad \mathbf{k}_1 . \mathbf{S}(\mathbf{k}_1) . \hat{\mathbf{z}} = 0.$$

Simple combinations of these conditions yield the off-diagonal elements as

$$\left. \begin{aligned} 2k_{1x} k_{1y} S_{xy} &= -k_{1x}^2 S_{xx} - k_{1y}^2 S_{yy} + k_{1z}^2 S_{zz}, \\ 2k_{1x} k_{1z} S_{xz} &= -k_{1x}^2 S_{xx} - k_{1z}^2 S_{zz} + k_{1y}^2 S_{yy}, \\ 2k_{1y} k_{1z} S_{yz} &= -k_{1y}^2 S_{yy} - k_{1z}^2 S_{zz} + k_{1x}^2 S_{xx}, \end{aligned} \right\} \quad (19)$$

which allows us to eliminate the off-diagonal elements in (17) and (18). Substituting (15)–(19) into (14) we have A_{zz}^N expressed in terms of the diagonal elements as follows:

$$A_{zz}^N = -2(\tfrac{1}{2}\pi)^{\frac{1}{2}} \int \frac{d\mathbf{k}_1}{(2\pi)^3} \int \frac{d\mathbf{k}_2}{4\pi} \frac{[\gamma_{zx} S_{xx}(\mathbf{k}_1) + \gamma_{zy} S_{yy}(\mathbf{k}_1) + \gamma_{zz} S_{zz}(\mathbf{k}_1)] E(k_2)}{v_0 k_2^2}, \quad (20a)$$

$$\gamma_{zx} \equiv \frac{k_1^2 - (\mathbf{k}_1 . \hat{\mathbf{k}}_2)^2}{(k_1^2 + k_2^2)^{\frac{1}{2}} k^2} \left[\tfrac{1}{2} k_z \left(\frac{k_{2y}}{k_{1y} k_{1z}} - \frac{k_{2x}}{k_{1x} k_{1z}} \right) \right] k_{1x}^2 + B_z^* \left[k_{2x}^2 + \left(\frac{k_{2y} k_{2z}}{k_{1y} k_{1z}} - \frac{k_{2x} k_{2z}}{k_{1x} k_{1z}} - \frac{k_{2x} k_{2y}}{k_{1x} k_{1y}} \right) k_{1x}^2 \right], \quad (20b)$$

$$\gamma_{zy} \equiv \frac{k_1^2 - (\mathbf{k}_1 . \hat{\mathbf{k}}_2)^2}{(k_1^2 + k_2^2)^{\frac{1}{2}} k^2} \left[\tfrac{1}{2} k_z \left(\frac{k_{2x}}{k_{1x} k_{1z}} - \frac{k_{2y}}{k_{1y} k_{1z}} \right) \right] k_{1y}^2 + B_z^* \left[k_{2y}^2 + \left(\frac{k_{2x} k_{2z}}{k_{1x} k_{1z}} - \frac{k_{2x} k_{2y}}{k_{1x} k_{1y}} - \frac{k_{2y} k_{2z}}{k_{1y} k_{1z}} \right) k_{1y}^2 \right], \quad (20c)$$

$$\gamma_{zz} \equiv \frac{k_1^2 - (\mathbf{k}_1 . \hat{\mathbf{k}}_2)^2}{(k_1^2 + k_2^2)^{\frac{1}{2}} k^2} \left[k_z k_{2z} - \tfrac{1}{2} k_z \left(\frac{k_{2x}}{k_{1x} k_{1z}} + \frac{k_{2y}}{k_{1y} k_{1z}} \right) k_{1z}^2 \right] + B_z^* \left[k_{2z}^2 + \left(\frac{k_{2x} k_{2y}}{k_{1x} k_{1y}} - \frac{k_{2x} k_{2z}}{k_{1x} k_{1z}} - \frac{k_{2y} k_{2z}}{k_{1y} k_{1z}} \right) k_{1z}^2 \right], \quad (20d)$$

$$B_z^* \equiv \left\{ k_z k_{1z} - k_z k_{2z} \frac{\mathbf{k}_1 . \mathbf{k}_2}{k_2^2} - 2 \frac{k_z^2}{k^2} [k_1^2 - (\mathbf{k}_1 . \hat{\mathbf{k}}_2)^2] \right\} (k_1^2 + k_2^2)^{-\frac{1}{2}} k^{-2}, \quad (20e)$$

where the $\mathbf{k}$ integration has been transformed into a $\mathbf{k}_2$ integration by $\mathbf{k} = \mathbf{k}_1 + \mathbf{k}_2$, and $\hat{\mathbf{k}}_2 \equiv \mathbf{k}_2/k_2$ is the unit vector along $\mathbf{k}_2$. The γ_{zi} arise from straightforward algebra when (19) is substituted into (15) and (16). We wish to reassure the reader that although γ_{zi} is complex looking, the integrations in (20a) can be performed quite simply, as is done in appendix A. Furthermore, the term $[k_1^2 - (\mathbf{k}_1 . \hat{\mathbf{k}}_2)]\, k_z k_{2z}$ is dominant in γ_{zz}, with the other terms of γ_{zz} providing a small correction. The terms $B^* k_{2x}^2$ and $B^* k_{2y}^2$ are dominant in γ_{zx} and γ_{zy}, respectively.

Equation (20a) can be expressed easily in terms of the stress $\langle u_i^2 \rangle$. To do so we write simply

$$S_{ii}(\mathbf{k}_1) \equiv S_{ii}(\mathbf{k}_1) \langle u_i^2 \rangle [(2\pi)^{-3} \int d\mathbf{k}_1 S_{ii}(\mathbf{k}_1)]^{-1},$$

$$E(k_2) \equiv E(k_2) (\tfrac{3}{2} v_0^2) [(4\pi)^{-1} \int d\mathbf{k}_2 E(k_2)]^{-1},$$

which are identities, to obtain

$$A_{zz}^N = -(k_{zz}^* v_0 \langle u_x^2 \rangle + k_{zy}^* v_0^2 \langle u_y^2 \rangle + k_{zz}^* v_0 \langle u_z^2 \rangle), \tag{21}$$

where k_{zi} is a wavenumber defined explicitly by

$$k_{zi}^* \equiv 3(\tfrac{1}{2}\pi)^{\frac{1}{2}} \int \frac{d\mathbf{k}_1}{(2\pi)^3} \int \frac{d\mathbf{k}_2}{4\pi} \frac{\gamma_{zi} E(k_2) S_{ii}(\mathbf{k}_1)}{k_2^2} \times \left[\int \frac{d\mathbf{k}_1}{(2\pi)^3} S_{ii}(\mathbf{k}_1) \int \frac{d\mathbf{k}_2}{4\pi} \frac{E(k_2)}{k_2^2} \right]^{-1}. \tag{21$'$}$$

That is, k_{zi}^* is the mean value of γ_{zi} averaged over the velocity spectra $S_{ii}(\mathbf{k}_1)$ and $E(k_2)\, k_2^{-2}$. The value of k_{zi}^* can be calculated readily, as is done in §4. As mentioned in I, $(k_{zi}^*)^{-1}$ is a novel kind of integral scale because it is a double integral over two spectra. This integral scale is a basic characteristic of the pressure–velocity correlation; a knowledge of k_{zi}^* is equivalent to a knowledge of A_{zz}^N.

To compare (20a) with the Rotta model we first express k_{si}^* in terms of the energy dissipation rate ϵ and the energy density $e_0 \equiv (\tfrac{3}{2}) v_0^2$. This is trivial to do because $(k_{zi}^* v_0)^{-1}$ and ϵ/e_0 both have the dimensions of time so that (20a) can be written immediately as

$$2A_{zz}^N \equiv -\frac{\epsilon}{e_0} [\beta_{zx} \langle u_x^2 \rangle + \beta_{zy} \langle u_y^2 \rangle + \beta_{zz} \langle u_z^2 \rangle], \tag{22}$$

where β_{zi} is a dimensionless proportionality constant given by

$$\beta_{zi} \equiv 2k_{zi}^* (e_0/\epsilon), \tag{23}$$

$$\beta_{zi} \equiv 6(\tfrac{1}{2}\pi)^{\frac{1}{2}} \left(\frac{v_0 e_0}{\epsilon} \right) \int \frac{d\mathbf{k}_1}{(2\pi)^3} \int \frac{d\mathbf{k}_2}{4\pi} \frac{\gamma_{zi} E(k_2) S_{ii}(\mathbf{k}_1)}{k_2^2 e_0 \langle u_i^2 \rangle}. \tag{24}$$

Equation (22) determines A_{zz}^N in terms of the stress $\langle u_i^2 \rangle$, and the numerical coefficients β_{zi} are determined in terms of measurable quantities (spectra). Note that (22) agrees with Rotta's model if $\beta_{zx} = \beta_{zy} = -\tfrac{1}{2}\beta_{zz}$. The numerical value of β_{zi} is calculated in §4 for two classes of spectral models. Expressions for the other diagonal pressure–strain-rate elements A_{xx}^N and A_{yy}^N are given in appendix B.

4. Theoretical calculation of β_{zi} and Rotta's constant

To complete the calculation of A_{zz}^N we must calculate the numerical values of the proportionality constants β_{zi}. This calculation is very similar to the previous calculation of C_{xz}, the Rotta constant, made in I (§6 and appendix B of I) for models of the spectrum $\mathbf{S}$. The sensitivity of β_{zi} to these models can be examined after the calculation.

The calculation of β_{zi} consists of performing the $\mathbf{k}_1$ and $\mathbf{k}_2$ integrations in (24). This integration is divided into two parts. First we integrate over the directions (spherical angles) of $\mathbf{k}_1$ and $\mathbf{k}_2$ and afterwards we integrate over their scalar magnitudes. The integrations over the directions of $\mathbf{k}_1$ and $\mathbf{k}_2$ are given in appendix A. There it is found, for spectra satisfying (29), that β_{zi} is given by (see (A 20), (A 28), and (A 29))

$$\left.\begin{aligned}\beta_{zi} &= d_i\,(\tfrac{1}{2}\pi)^{\frac{1}{2}}\left(\frac{v_0 e_0}{\epsilon}\right)\int_0^\infty dk_1\int_0^\infty dk_2\,\frac{k_1^2 k_2^2 E(k_2)\,E_{ii}(k_1)}{(k_1^2+k_2^2)^{\frac{3}{2}}\,e_0\langle u_z^2\rangle},\\ d_x &= d_y = -0{\cdot}36,\quad d_z = 0{\cdot}72,\end{aligned}\right\} \tag{25}$$

where $E_{ii}(k_1)$ is a scalar spectrum obtained by integrating $S_{ii}(\mathbf{k}_1)$ over a spherical shell of radius k_1; i.e.

$$E_{ii}(k_1) \equiv \frac{k_1^2}{4\pi}\int_0^{2\pi} d\phi_1\int_0^{\pi} d\theta_1 \sin\theta_1\, S_{ii}(\mathbf{k}_1), \tag{26}$$

which satisfies $\int_0^\infty dk E_{ii}(k) = \langle u_i^2\rangle$. Comparing (26) with (12) it is seen that E_{ii} also satisfies $E = \frac{1}{2}(E_{xx}+E_{yy}+E_{zz})$. The uncertainties in our values of d_x, d_y, d_z are small for the model spectra (29) that are used in this section. The variation of d_i with variations of the model is calculated in §4.3.

Before integrating (25) to obtain the numerical value of β_{zi}, we call attention to the expression obtained by substituting (25) into (22):

$$2A_{zz}^N = -0{\cdot}72(\tfrac{1}{2}\pi)^{\frac{1}{2}}\,v_0\int_0^\infty dk_1\int_0^\infty dk_2\,\frac{k_1^2k_2^2E(k_2)}{(k_1^2+k_2^2)^{\frac{3}{2}}\,e_0}\,[E_{zz}(k_1)-\tfrac{1}{2}E_{xx}(k_1)-\tfrac{1}{2}E_{yy}(k_1)]. \tag{27}$$

(Similar expressions for A_{xx}^N and A_{yy}^N are given in appendix B.) There are two evident features of (27): A_{zz}^N is seen to approach zero (as it must) as the spectrum approaches isotropy and, more noteworthy, A_{zz}^N can be non-zero (with a consequent flow of fluid velocity) even when the fluid stress is isotropic ($\langle u_x^2\rangle = \langle u_y^2\rangle = \langle u_z^2\rangle$). This is because the spectrum need not be isotropic even though the stress $\langle \mathbf{uu}\rangle$ is; i.e. $E_{zz}-\frac{1}{2}E_{xx}-\frac{1}{2}E_{yy}$ need not be zero even though $\langle u_z^2\rangle-\frac{1}{2}\langle u_x^2\rangle-\frac{1}{2}\langle u_y^2\rangle$ is zero. Such a case cannot be described by Rotta's model. We refer to that case as a higher-moment spectral anisotropy ($\langle \mathbf{uu}\rangle$ is the zeroth moment of the spectrum $\mathbf{S}$), and will discuss it in §4.3. For the remainder of this section we restrict ourselves to calculating β_{zi} for the more typical case of anisotropic fluid stress $\langle \mathbf{uu}\rangle$.

4.1. *Calculation of β_{zi} for the zeroth-moment model (Rotta's model)*

The numerical value of β_{zi} is obtained by integration in (25). However, to perform this integration we must resort to a model of E and E_{ii}. Afterwards we will examine the sensitivity of β_{zi} to that model. We use the same model of E that was used in I to calculate the off-diagonal Rotta constant C_{xz} and that was previously used by Comte-Bellot & Corrsin (1966) and by Reynolds (1976) to estimate a parameter of decaying turbulence. It is given by $E(k) = \alpha\epsilon^{\frac{2}{3}}k^{-\frac{5}{3}}$ for $k_L \leqslant k \leqslant k_\nu$, $E(k) = \alpha\epsilon^{\frac{2}{3}}(k_L^{-\frac{5}{3}-m})\,k^m$ for $k \leqslant k_L$, and $E(k) \simeq 0$ for $k > k_\nu$, where k_ν is the 'cut-off' wavenumber due to molecular viscosity, $m > -1$ is an adjustable parameter, and $\alpha \simeq 1{\cdot}5$ is the Kolmogoroff constant. In this model, there is made the familiar assumption that $E(k)$ has a maximum value, or peak, at some wavenumber k_L, and that the main contribution to the velocity integral $\int_0^\infty dkE(k)$ comes from k in the vicinity of k_L. This vicinity is called the energy-containing region. The relationship between ϵ and v_0 or e_0 for this model is given by substitution of E into (12′):

$$\left.\begin{aligned} v_0^2 &= \alpha\epsilon^{\frac{2}{3}}k_L^{-\frac{2}{3}}[1+\tfrac{2}{3}(m+1)^{-1}-R_\nu^{-\frac{1}{2}}] \quad (R_\nu \gg 1),\\ R_\nu &\equiv (k_\nu/k_L)^{\frac{4}{3}}, \end{aligned}\right\} \tag{28}$$

where the 'Reynolds-number' term $R_\nu^{-\frac{1}{2}}$ comes from the viscous 'cut-off' at k_ν. R_ν is related to the viscosity ν by $R_\nu \approx v_0/\nu k_L$.

The modelling of E_{ii} is a little intricate because, as seen in (27), A_{zz}^N depends on the differences between E_{xx}, E_{yy}, and E_{zz}. To aid us in understanding the influence of E_{ii} on A_{zz}^N (and β_{zi}) it is quite useful to characterize the form or shape of E_{ii} in terms of moments. Thus we define the nth moment of E_{ii} by $\int_0^\infty dkk^nE_{ii}(k)$. The zeroth moment is simply the stress, i.e. $\int_0^\infty dkE_{ii} \equiv \langle u_i^2\rangle$. We will consider two models for our calculation of β_{zi}: first, the elementary model in which E_{xx}, E_{yy}, E_{zz} differ from each other only in their zeroth moment but not in their higher moments. That model is simply

$$\frac{E_{xx}(k)}{\langle u_x^2\rangle} = \frac{E_{yy}(k)}{\langle u_y^2\rangle} = \frac{E_{zz}(k)}{\langle u_z^2\rangle} \quad \text{(spectral model 1)}, \tag{29}$$

which determines $E_{ii}(k)$ to be $E_{ii}(k) = [\langle u_i^2\rangle/e_0]E(k)$ when use is made of $E = \frac{1}{2}(E_{xx}+E_{yy}+E_{zz})$. This model is of special interest because, as seen below, it leads trivially to Rotta's relation (30). The zero-moment model is so-named because it guarantees that the zero moments are correct $[\int_0^\infty dkE_{ii}(k) = \langle u_i^2\rangle]$ – which is important. The model is not numerically correct at very large k or at very small k, but it is accurate for the intermediate range of k wherein is found a large contribution to the integrations in (25). The most questionable feature of this model is that the peaks of E_{xx}, E_{yy}, and E_{zz} all occur at the same wavenumber k_L. (This feature is tested in §4.3.) This model also deviates a little from local isotropy. However, small deviations have been found in shear flows (Champagne *et al.* 1970). The second model we will use for E_{ii} contains differences between the higher moments of $E_{xx}/\langle u_x^2\rangle$, $E_{yy}/\langle u_y^2\rangle$, and $E_{zz}/\langle u_z^2\rangle$, including different peak wavenumbers. This higher-moment model is given in §4.3.

It is easily seen that the model (29) gives us Rotta's relation. That is, substitution of (29) in (25) gives immediately $\beta_{zx} = \beta_{zy} = -\frac{1}{2}\beta_{zz}$ so that the pressure–strain rate (22) becomes

$$\left.\begin{aligned} 2A_{zz}^N &= -\frac{\epsilon}{e_0}C_{zz}(\tfrac{2}{3}\langle u_z^2\rangle - \tfrac{1}{3}\langle u_x^2\rangle - \tfrac{1}{3}\langle u_y^2\rangle)\\ C_{zz} &\equiv \tfrac{3}{2}\beta_{zz}, \end{aligned}\right\} \tag{30}$$

which is of the same form as Rotta's model of A_{zz}^N since $[\frac{2}{3}\langle u_z^2\rangle - \frac{1}{3}\langle u_x^2\rangle - \frac{1}{3}\langle u_y^2\rangle] = \langle u_z^2\rangle - \frac{2}{3}e_0$. Note that, for later convenience, we have defined the (Rotta) constant $C_{zz} \equiv \frac{3}{2}\beta_{zz}$ to make (30) conform with the notation in I. Similarly it is found in appendix B that the other diagonal elements of the pressure–strain rate also have the form of Rotta's model:

$$2A_{ii}^N = -\frac{\epsilon}{e_0}C_{ii}(\langle u_i^2\rangle - \tfrac{2}{3}e_0) \quad (i = x, y, z),$$

$$C_{xx} = C_{yy} = C_{zz} \quad \text{(for model (29))}.$$

The C_{ii} are equal for the model spectrum in (29), but not necessarily for other spectra.

Finally, the numerical value of the Rotta constant $C_{zz} \equiv \frac{3}{2}\beta_{zz}$ is obtained by substituting the model expressions of E_{ii} into (25) and integrating:

$$\left.\begin{aligned} C_{zz} &= (\tfrac{3}{2})^2 (\tfrac{1}{2}\pi)^{\frac{1}{2}} (0{\cdot}74)\, \alpha^{\frac{3}{2}} \frac{[1+\frac{2}{3}(m+1)^{-1}]^{\frac{3}{2}}}{1+1{\cdot}3(m+1)^{-1}} (1+R_\nu^{-\frac{3}{4}} - R_\nu^{-\frac{1}{4}})(1-R_\nu^{-\frac{1}{2}}), \\ C_{zz} &= 2{\cdot}1 \quad (\text{for } R_\nu = 50), \\ C_{zz} &= 2{\cdot}9 \quad (\text{for } R_\nu = 1{,}000), \\ C_{zz} &= 3{\cdot}6 \quad (\text{for } R_\nu = \infty), \end{aligned}\right\} \tag{31}$$

which can be shown to be insensitive to m, the large-wavelength behaviour of the spectrum. Such an insensitivity was also found in I for the off-diagonal constant C_{xz}. A new, and quite interesting, feature found in (31) is that C_{zz} varies significantly with the 'Reynolds number' R_ν even when R_ν is large. Consequently, C_{zz} may vary from one flow to another in a predictable way. The numerical value of theoretical C_{zz} lies within the range deduced by empirical and experimental determinations (e.g. Hanjelic & Launder 1972; Launder *et al.* 1975; Champagne *et al.* 1970; Reynolds 1976) but is substantially larger than the value deduced by Lumley & Newman (1977) from the data of Comte-Bellot & Corrsin (1966) (a value, incidentally, which implies slow return to isotropy when near isotropy). We will discuss the variously determined values of C_{zz}, and the uncertainty in our C_{zz} caused by our assumptions, in §6. First, we wish to call attention to an inadequacy of Rotta's model. This inadequacy is in the assumption that $C_{xz} = C_{zz}$; i.e. that the coefficient of the off-diagonal element (of the pressure–strain-rate model) is equal to the coefficient of the diagonal element in the linear limit of very small anisotropy. There is a qualitative and numerical difference between C_{xz} and C_{zz}, as discussed next.

4.2. *Comparison between C_{zz} and C_{xz}: an inadequacy of Rotta's model*

While substantial deviations from Rotta's model are known to occur when the anisotropy is large, it is widely supposed that the Rotta model is valid in the asymptotic limit of small anisotropy and large Reynolds number. However, we find that this is not the case; a numerical and qualitative difference exists between C_{zz} and C_{xz} even for vanishing small anisotropy. This difference is easily seen by comparison of (25) with the expression for C_{xz} given by (39) of I. The latter expression is

$$C_{xz} = (\tfrac{1}{2}\pi)^{\frac{1}{2}} \left(\frac{v_0 e_0}{\epsilon}\right) \int_0^\infty dk_1 \int_0^\infty dk_2 \frac{k_1^2 k_2^2 E(k_2)\, E_{xz}(k_1)}{(k_1^2+k_2^2)^{\frac{3}{2}}\, e_0 \langle u_x u_z\rangle}, \tag{32}$$

where $E_{xz}(k_1) \equiv (2\pi)^{-3} k_1^2 \int_0^{2\pi} d\phi_1 \int_0^\pi d\theta_1 \sin\theta_1 S_{xz}(\mathbf{k}_1)$ is the average of $S_{xz}(\mathbf{k}_1)$ over a spherical shell just as $E_{zz}(k_2)$ is the average of $S_{zz}(\mathbf{k}_1)$ over a spherical shell. The ratio of this C_{xz} to $C_{zz} = \frac{3}{2}\beta_{zz}$ in (25) is

$$\frac{C_{xz}}{C_{zz}} = \frac{2\int_0^\infty dk_1 \int_0^\infty dk_2\, k_1^2 k_2^2 (k_1^2+k_2^2)^{-\frac{3}{2}} E(k_2)\, E_{xz}(k_1)/\langle u_x u_z\rangle}{3d_z \int_0^\infty dk_1 \int_0^\infty dk_2\, k_1^2 k_2^2 (k_1^2+k_2^2)^{-\frac{3}{2}} E(k_2)\, E_{zz}(k_1)/\langle u_z^2\rangle}. \tag{33}$$

The qualitative difference between C_{xz} and C_{zz} comes from the fact that E_{xz} decreases rapidly with k_1 in the inertial range (i.e. it is found experimentally that $E_{xz}(k_1) \propto k_1^{-\frac{7}{3}}$ or $k_1^{-\frac{8}{3}}$ when $k_1 \gg k_L$), whereas $E_{zz}(k_1)$ decreases relatively slowly with k_1 in the inertial range (i.e. $E_{zz}(k_1) \propto k_1^{-\frac{5}{3}}$ when $k_1 \gg k_L$). Consequently, the contribution to C_{xz} from inertial-range scales is very small, whereas the contribution to C_{zz} from inertial-range scales is large. Thus, C_{xz} and C_{zz} differ physically.

There is also a significant numerical difference between C_{xz} and C_{zz}, as seen by comparing (31) with equation (44) of I: The numerical ratio between them is

$$\frac{C_{zz}}{C_{xz}} = 2\cdot1\,(1+R_\nu^{-\frac{3}{4}}-R_\nu^{-\frac{1}{4}}) \quad \text{(for model (29))}, \tag{34}$$

where we have multiplied C_{xz} in equation (44) of I by the factor $(1-R_\nu^{-\frac{1}{4}})$ which was previously omitted. Thus C_{zz} is about twice as large as C_{xz}. This difference can be attributed to inertial-range scales (as seen by analysing the integrations in (33) with $\frac{3}{2}d_z \simeq 1$).

One could argue that the assumptions used to derive C_{xz} and C_{zz} may vitiate (33). However, the same assumptions were used for C_{xz} as for C_{zz}, and the resulting errors tend to cancel out. Such a cancellation is especially evident for the main assumption of this work – the cumulant discard. In this assumption we derive (in I) a third-order velocity correlation as a product of two second-order correlations multiplied by a characteristic time $\tau^* \simeq (kv_0)^{-1}$ (i.e. $\langle vvv\rangle \propto \int d\mathbf{k}\langle vv\rangle\langle vv\rangle\tau^*$), and this τ^* cancels out of the ratio (33). More importantly, there is a fundamental basis for the form of (32) and (25) as follows. From the point of view of elementary perturbation theory (non-linear interactions between Fourier components of p and $\partial u_x/\partial z$) the off-diagonal pressure–strain-rate element A_{xz}^N can be viewed as the interaction (coupling) between the turbulence kinetic energy $E(k)$ and the off-diagonal velocity correlation S_{xz} (e.g. $\partial u_x/\partial z \rightarrow ik_z u_x \rightarrow -ik_z \mathbf{u}\,.\,\nabla u_x \propto k_z^2 u_x u_z$, and $p \propto \mathbf{u}\,.\,\mathbf{u}$). For this reason, the magnitude of A_{xz}^N depends on the product $(E)\,(S_{xz})$ summed over all Fourier components. Similarly, A_{zz}^N represents the coupling between the turbulence energy E and diagonal velocity correlations S_{ii}, so that $A_{zz}^N \propto (\mathrm{E})\,(S_{ii})$. Consequently, A_{zz}^N differs from A_{xz}^N because of the intrinsic difference between S_{ii} and S_{xz}. Furthermore, A_{zz}^N (and C_{zz}) obtains a greater contribution from inertial-range scales than does A_{xz}^N (and C_{xz}) because, in that range, S_{zz} is much greater than S_{xz}. Hence, the fundamental difference between C_{xz} and C_{zz} exists regardless of the degree of anisotropy.

4.3. *Variations of A_{ii}^N and C_{ii} with spectra E_{ii}: higher-moment model*

In §4.2, A_{zz}^N and C_{zz} were calculated for the model spectrum (29). The purpose of this section is to examine the variations of A_{zz}^N as well as of A_{xx}^N and A_{yy}^N caused by variations in the spectrum; i.e. to examine the model-dependence of A_{ii}^N. For this purpose we choose a simple spectral model in which the maximum of E_{xx} occurs at a wave-number k_L which differs from the wavelength k_L' at which E_{zz} is a maximum (to simplify the calculation we take $E_{yy} = E_{zz}$, although $\langle u_y^2\rangle > \langle u_z^2\rangle$ in a simple shear flow). This model is illustrated in figure 1, and is suggested by the data of Kaimal *et al.* (1972). Our purpose is to determine how C_{ii} and A_{ii}^N vary with k_L/k_L'; we characterize k_L'/k_L by the first-moment expression

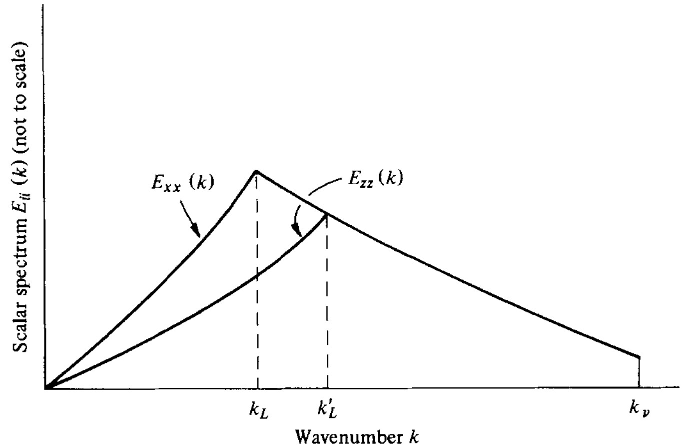

FIGURE 1. Higher-moment spectral model of $E_{xx}(k)$ and $E_{zz}(k)$ showing that $k'_L \neq k_L$.

$$\frac{k'_L}{k_L} = \frac{\int_0^\infty dk\, kE_{zz}(k)/\langle u_z^2\rangle}{\int_0^\infty dk\, kE_{xx}(k)/\langle u_z^2\rangle}. \tag{35}$$

(As was pointed out by a referee A_{ii}^N could vary with $\langle u_z^2\rangle/\langle u_y^2\rangle$ as well as with k'_L/k_L. To simplify matters, we concentrate on the variation of A_{ii}^N with k'_L/k_L, which we have found to be a greater variation.) In our model, we also take $\langle u_z^2\rangle/\langle u_x^2\rangle = (k_L/k'_L)^{\frac{2}{3}}$, which is suggested by both experimental and theoretical considerations discussed in appendix C, so that in this model k_L/k'_L is not an independent parameter. The details of the calculation of A_{ii}^N are given in appendix C. There we find very large deviations from Rotta's model when $k'_L/k_L \geqslant 2$. Particularly unexpected, is the discovery that $\beta_{zx} \neq -\frac{1}{2}\beta_{zz}$ in (22), so that the basic form of Rotta's model is violated. Thus, for $k'_L/k_L = 2$, $\langle u_z^2\rangle/\langle u_x^2\rangle = 2^{-\frac{2}{3}}$ we obtain

$$\left.\begin{aligned} 2A_{zz}^N &= -\frac{\epsilon}{e_0}C^0[1{\cdot}17\langle u_z^2\rangle - 0{\cdot}26\langle u_x^2\rangle - 0{\cdot}58\langle u_y^2\rangle],\\ 2A_{xx}^N &= -\frac{\epsilon}{e_0}C^0[0{\cdot}52\langle u_x^2\rangle - 0{\cdot}58\langle u_y^2\rangle - 0{\cdot}58\langle u_z^2\rangle],\\ 2A_{yy}^N &= -\frac{\epsilon}{e_0}C^0[1{\cdot}17\langle u_y^2\rangle - 0{\cdot}26\langle u_x^2\rangle - 0{\cdot}58\langle u_z^2\rangle],\end{aligned}\right\} \tag{36}$$

where $C^0 \simeq 2$. Equation (36) shows a surprisingly large and fundamental deviation from Rotta's model. Indeed, this equation shows that the pressure–strain rate is not even proportional to the stress anisotropy $\langle \mathrm{u}_i^2\rangle - \frac{2}{3}e_0$, when k'_L/k_L equals 2. Note, particularly, that the coefficients 0·52 in A_{xx}^N and 0·26 in A_{zz}^N are very large deviations. In fact, the deviations can be so large that even the sign of A_{ii}^N can differ from Rotta's law. Such a difference in sign from Rotta's model was recently found in atmospheric boundary-layer flow (Wyngaard 1980). In such a case, of course, nonlinear terms are essential to describe A_{ii}^N, and this will be discussed at length in a companion paper. A physical interpretation of (36) is discussed at the end of appendix C. Briefly, it is that if the shapes of the spectra $E_{xx}(k)/\langle u_x^2\rangle$ and $E_{zz}(k)/\langle u_z^2\rangle$ differ from each other, then some of the energy that would otherwise be transferred from E_{xx} to E_{zz} is used, instead, to redistribute the energy contained within E_{xx} (i.e. to change the shape of E_{xx}). In general, if $k'_L \neq k_L$, then A_{ii}^N is given by

$$2A_{ii}^N = -\frac{\epsilon}{e_0} C^0 [\beta_{ix}\langle u_x^2\rangle + \beta_{iy}\langle u_y^2\rangle + \beta_{iz}\langle u_z^2\rangle], \tag{37}$$

where the coefficients $\beta_{ij} \equiv \beta_{ij}(k'_L/k_L)$ are functions of k'_L/k_L, and their numerical values are given explicitly by (C 4), (C 6)–(C 9). These coefficients approach the Rotta values $\beta_{ii} = 1$, $\beta_{xy} = \beta_{xz} = \beta_{yz} = -\frac{1}{2}$, etc. when k'_L/k_L approaches unity; i.e. the Rotta form holds in the asymptotic limit of small spectral anisotropy. For shear flows, we estimate (from the assumed dependence of β_{ij} on $\langle u_z^2\rangle/\langle u_x^2\rangle \simeq (k_L/k'_L)^{\frac{2}{3}}$) that $\langle u_z^2\rangle$ must be within 20 % of $\langle u_x^2\rangle$ in order for the Rotta form to be correct within a factor of 2 – which is still a substantial deviation. We conclude that a term linear in the stress anisotropy a_{ii} as proposed by Rotta does not always exist (such a term is approached asymptotically only as the spectral anisotropy approaches zero, or when $E_{ii}/\langle u_i^2\rangle$ is the same for all i). This is because the linear term of A_{ii}^N is actually linear in the *spectral* anisotropy S_{ii}^A, rather than in the *stress* anisotropy a_{ii}, and a small value of a_{ii} does not always imply a small value of S_{ii}^A. Another way to explain this conclusion is to point out that S_{xx}^A can itself be a nonlinear function of a_{ii}, so that a quantity linear in S_{ii}^A ($2A_{ii}^N$ is such a quantity) is not always linear in a_{ii}.

Generally speaking, S_{ii}^A requires two or more flow parameters to specify its influence on A_{ii}^N. The stress anisotropy a_{ii} furnishes only one; k'_L/k_L furnishes another. To illustrate this symbolically we note that A_{zz}^N is proportional to $S_{ii}^A(k)$ and the latter, in our model, can be expressed in the form

$$S_{ii}^A(k) = a_{ii}F_1(k) + F_{ii}(k'_L/k_L, k),$$

where $F_1(k)$ is a function normalized to unity, and F_{ii} is a function which depends on k'_L/k_L (but not on a_{ii}) and vanishes only when k'_L/k_L equals unity. For this model, $S_{ii}^A(k)$ cannot be approximated by the Rotta form $a_{ii}F_1(k)$ unless k'_L/k_L is close to unity, which illustrates why the validity of Rotta's model is more restricted than is generally supposed. (Our extreme example is that it is possible for a_{ii} to be zero and yet for S_{ii}^A, and consequently A_{ii}^N, to be non-zero. A model based on $\mathbf{a}$ is obviously inadequate for that case. In more general cases, S_{ij}^A/a_{ij} is not necessarily the same for all i and j, and the differences cause deviations from Rotta's model.)

The important question that emerges is whether or not k'_L/k_L differs from unity in real flows; i.e. do higher moments of E_{xx} differ from those of E_{zz} in actual flows? This question can be answered by experimental determinations of $E_{ii}(k)$ (which require measurements of $S_{ii}(\mathbf{k})$ as a function of $\mathbf{k}$). Information on $S_{xx}(\mathbf{k})$ has been partly provided by experiments (e.g. the spatial isocorrelation measurements of Champagne *et al.* 1970 and Harris *et al.* 1977). However, further measurements of S_{xx} and of S_{zz} and S_{yy} are required to determine k'_L/k_L. The atmospheric measurements of $S_{ii}(k_x)$ by Kaimal *et al.* (1972, figure 17) suggest that k'_L/k_L is quite large (exceeding 3) at very-high-Reynolds-number shear flow. In that case, one can expect large deviations from Rotta's model as in (36). Again, we emphasize that these deviations occur in the *linear* anisotropy term – nonlinear terms have not been considered. We also wish to emphasize our belief that nonlinearities can be very important (e.g. Harris *et al.* 1977), and these will be discussed separately. Indeed, the observations of Harris *et al.* cannot be explained by only the linear term (37).

5. Further discussion of invalidities of Rotta's model

5.1. *Diagonal elements of Rotta's model*

The purpose of this section is to discuss and emphasize the limitations of Rotta's model noted in §§4.2 and 4.3. There, it was seen that Rotta's model is not generally valid as a linear anisotropy term. The main deficiency is that the diagonal pressure–strain-rate element A^N_{ii} is not even proportional to the stress anisotropy a_{ii} except for special spectra such as $E_{xx}/\langle u^2_x\rangle = E_{yy}/\langle u^2_y\rangle = E_{zz}/\langle u^2_z\rangle$ (only the zeroth moments of E_{ii} differ). The other reason is that $C_{xz} \neq C_{ii}$ (although C_{xz}/C_{ii} is determined). In both cases, the deficiencies of Rotta's model occur because A^N_{ii} depends on small spectral scales (the inertial-subrange scales) as well as on the larger scales (the energy-containing scales), and it is not generally possible to characterize simultaneously both spectral ranges with a single flow quantity such as the stress anisotropy a_{ii}. Put simply, more than one moment of the spectrum may be required to characterize the large spectral range that determines A^N_{ii} (a_{ii} furnishes only one such moment). We must therefore conclude that Rotta's hypothesis is not generally valid as a linear term in anisotropy. It can be used as such for special spectra when account is taken of C_{xz}/C_{ii}.

Curiously enough, although A^N_{ii} is not generally linear in the stress aniostsropy a_{ii}, (27) suggests that Rotta's physical arguments work very well for the spectra $E_{ii}(k_1)$ at each wavenumber (as distinct from the integral over all wavenumbers). That is, it is seen that $2A^N_{zz}$ depends on the spectra in the anisotropic form

$$E^A_{zz}(k_1) \equiv E_{zz}(k_1) - \tfrac{1}{2}E_{xx}(k_1) - \tfrac{1}{2}E_{yy}(k_1).$$

Hence, Rotta's physical arguments are borne out at each particular wavenumber. For this reason, if all the energy were concentrated in a single wavenumber, described by $E_{ii}(k) = \langle u^2_i\rangle\,\delta(k-k_0)$, we would then have $2A^N_{zz}$ proportional to $\langle u^2_z\rangle - \tfrac{1}{2}\langle u^2_x\rangle - \tfrac{1}{2}\langle u^2_y\rangle$, as hypothesized by Rotta. The difficulty with this hypothesis, we see, is that different scales are weighted differently in their contribution to $2A^N_{zz}$ because $E^A_{zz}(k_1)$ is not generally proportional to $\langle u^2_z\rangle - \tfrac{1}{2}\langle u^2_x\rangle - \tfrac{1}{2}\langle u^2_y\rangle$. Consequently, Rotta's arguments are qualitatively correct but not generally quantitative.

5.2. *Off-diagonal elements A^N_{xz}*

The second deficiency found in Rotta's model is that $C_{xz} \neq C_{ii}$. This difference occurs, as explained in §4.2, because C_{xz} depends on E_{xz} and consequently receives important contributions from only large wavelengths (of E_{xz}), whereas C_{ii} depends on E_{ii} and consequently receives important contributions from both small and large wavelengths (of E_{ii}). The ratio C_{ii}/C_{xz} is computed to be about 2 for the zeroth-moment spectral model. For general spectra, A^N_{xz} can be modelled by a single flow quantity, whereas A^N_{ii} cannot. In all cases we conclude that the off-diagonal elements of the pressure–strain rate should be modelled differently from the diagonal elements.

6. Comparisons with experiments

In this section we will briefly compare our theory with the experiments of Comte-Bellot & Corrsin (1966) and Champagne *et al.* (1970). We also wish to comment on the very large nonlinearity observed by Harris *et al.* (1977). The comparisons should necessarily be brief because, for reasons given in §§4.3 and 5, a calculation of A_{ii}^N requires more information about the spectra E_{ii} or S_{ii} than is available.

The data of Comte-Bellot & Corrsin was used with the Rotta form of A_{11}^N by Lumley & Newman (1977) to derive $C_{11} = 1$ for the Rotta constant ($C_{11} = 2$ with their definitions); a value, it was pointed out, that could explain the observed slow return to isotropy. This value differs from (31) and was observed in I to differ from our theoretically calculated value $C_{xz} = 1{\cdot}6$. Several possible explanations for this difference were mentioned. Among these is that our theory is limited to quasi-stationarity, whereas the experiments were for rapid energy decay; a second possibility is that the cumulant-discard approximation of the theory introduces a numerical error. These explanations are still possibilities to be investigated. The third possible reason given seems to be borne out by our present calculation: this is that the Rotta model may not be correct even when **a**, the stress anisotropy, is small as in the experiments. Thus, we have shown that the validity of Rotta's model requires spectra of a special form, and the experiments of Comte-Bellot & Corrsin (1966) do not provide information on S_{ii} needed to determine whether or not Rotta's form is valid for that experiment. In a decay experiment there is no obvious source of large-scale wavelengths that can ensure that the shapes of $E_{yy}/\langle u_y^2\rangle$ and $E_{xx}/\langle u_x^2\rangle$ are the same. This may explain why departure from isotropy at moderate as well as large correlation distances is found in the later grid-decay experiments of Comte-Bellot & Corrsin (1971). Curiously enough, if we were to assume (although we do not have the right to) that the ratio of spectral peak wavelengths is given by $k_L'/k_L \simeq (\langle u_x^2\rangle/\langle u_z^2\rangle)^{\frac{3}{2}}$ as in our shear-flow model of §4.3, then we find using (37) that the stress term $\beta_{11}\langle u_1^2\rangle + \beta_{12}\langle u_2^2\rangle + \beta_{13}\langle u_3^2\rangle$ equals approximately $\frac{1}{2}a_{11}$ instead of the Rotta value a_{11}. Use of that value (i.e. replacement of a_{11} by $\frac{1}{2}a_{11}$) in Lumley & Newman's (1977) expression for $2A_{11}^N$ would yield $C_{11} = 2$ (twice what they obtained), which is also in closer agreement with the values calculated in §4.1. The observed slow return to isotropy might then be explained by the fact that there is no energy source or other mechanism to isotropize the largest scales. However, we must emphasize that this discussion is entirely based on conjecture since $E_{ii}/\langle u_i^2\rangle$ is not known sufficiently, and, consequently, $2A_{11}^N$ cannot be determined precisely. Consequently, the interpretation of Lumley & Newman remains a possibility.

The experiments of Champagne *et al.* have the Rotta constants C_{ii} given by $C_{xx} \simeq 2{\cdot}3$, $C_{zz} \simeq 1{\cdot}8$, $C_{yy} \simeq 3{\cdot}0$, after multiplying by $\frac{3}{2}$ to make their definition of C_{ii} agree with ours. The average of these values is in good agreement with (31). However, this average agreement is somewhat fortuitous because, for a shear flow, the zero-moment model on which (31) is based does not rigorously apply. Equation (37) should be used instead. In addition, nonlinear anisotropic corrections (or corrections of order $(\partial U/\partial z)^2$) must be suspected because of the large difference between the experimental C_{zz} and C_{yy}. The difference demonstrates that, even for fairly weak turbulence, the deviations from Rotta's model are significant. This conclusion is not offset by inclusion of the mean-strain-rate contribution A_{ii}^M because it does not make a linear contribution to $2\rho_0^{-1}\langle p\,\partial u_i/\partial i\rangle$; the linear anisotropic part of A_{ii}^M is zero.

A dramatic observation of deviations from Rotta's model was made by Harris *et al.* (1977). They measured $2A_{ii}^N(e_0/\epsilon)\, a_{ii}^{-1}$, which we denote by C_{ii}^* for convenience, and found that C_{yy}^* was large and 'highly variable', ranging from 4 to 12 within a section of their apparatus, while the observed values of C_{xx}^* and C_{zz}^* were much smaller and relatively constant. We point out a simple fact about this curious behaviour. This fact is that the values observed for C_{yy}^* are large because the linear anisotropy $a_{yy} \equiv \langle u_y^2\rangle - \frac{2}{3}e_0$ is markedly small in those experiments. The influence of a small a_{yy} on C_{yy}^* is easily seen if one writes the total contribution to the pressure–strain rate A_{yy}^N in the form

$$2A_{yy}^N = -\frac{\epsilon}{e_0}\,(C_{yy}a_{yy} + \Delta_{yy}), \tag{38}$$

where $C_{yy}a_{yy}$ is the Rotta term and Δ_{yy} is the deviation from the Rotta term. Clearly, if, as in the experiment, a_{yy} is very small, then measurements of $2A_{yy}^N e_0/\epsilon a_{yy}$ will be very large ($\gg C_{yy}$), even if the deviation Δ_{yy} is not large. That this is, indeed, the experimental situation of Harris *et al.* is found from the measurements of $\langle u_i^2\rangle$ given in their figure 3. For example, at the downstream distance $x_1/h = 10$, they have $u_x^2/U_0^2 = 24\times 10^{-4}$, $u_y^2/U_0^2 = 14\times 10^{-4}$, $u_z^2/U_0^2 = 9\times 10^{-4}$ so that $a_{xx} = 8{\cdot}3\times 10^{-4}U_0^2$, $a_{yy} = -1{\cdot}7\times 10^{-4}U_0^2$, and $a_{zz} = -6{\cdot}7\times 10^{-4}U_0^2$. It is seen that $|a_{yy}|$ is much smaller than a_{xx} and $|a_{zz}|$, with the consequence that $|2A_{ii}^N e_0/\epsilon a_{ii}|$ is much larger for $i = y$ than for $i = x$ and z (since $|A_{yy}^N|$ is nearly equal to $|A_{zz}^N|$ and $\frac{1}{2}|A_{xx}^N|$ in the experiment). Hence, even a moderate nonlinearity can cause a large deviation from Rotta's model when a_{yy} is small.

7. Errors of the calculations

The major sources of error or uncertainties in our calculation are: (*a*) the simplifications and assumptions about the spectra used in the angular integrations given in appendix A; (*b*) the assumption of an inertial subrange at large k; and (*c*) the cumulant neglect used to derive (9). The errors due to spectral simplifications and approximations were discussed in detail in §7 of I and estimated to be only a few per cent for a given model of $E(k)$ and $E_{xz}(k)$. That discussion and conclusion applies to our calculation in appendix A and §4. It is the deviation of the spectra $E_{ii}(k)$ from the assumed inertial subrange that can cause significant variations of C_{ii} or A_{ii}^N. That is, the calculated values of C_{ii} and A_{ii}^N can be quite different if E_{ii} does not vary as $k^{-\frac{5}{3}}$ in the subrange ($k_L < k < k_\nu$). Indeed, the existence of an inertial subrange is the basis of our calculation of the numerical values of C_{ii} and β_{ij} in §4.

The major uncertainty of the theory is the neglect of the two-time fourth-order velocity cumulant, which is not to be confused with the neglect of a one-time cumulant in quasi-normal theory (e.g. Proudman & Reid 1954). As mentioned in I, the error caused in A_{ii}^N or C_{ii} by our cumulant neglect has not yet been estimated – although this may be done at a future time.

8. Summary and conclusions

(1). (*a*) The (turbulent part of) the off-diagonal pressure–strain-rate elements $2A_{ii}^N$ are derived theoretically in terms of measurable quantities (velocity spectra **S**). The derived expression, which is given by (9), is valid to general order in the anisotropy.

(*b*) The theoretical pressure–strain rate is then evaluated explicitly in terms of the Reynolds stress $\langle u_i^2 \rangle$ to first order in anisotropy, and the numerical (Rotta-type) constants) are derived in terms of the spectrum S_{ii} in §3.

(2). It is proved that the diagonal element $2A_{ii}^N$ is proportional to the stress anisotropy a_{ii} (in agreement with the Rotta relation) provided that the spectra has a special form such as the zeroth-moment model; otherwise it is not. For that spectral model, the Rotta constants C_{ii} are calculated theoretically (and given by (31)). It is shown that C_{ii} is quite insensitive to the long-wavelength behaviour of the spectrum, but that C_{ii} varies with the 'Reynolds number' R_ν even for large R_ν.

(3). The diagonal elements $2A_{ii}^N$ are calculated explicitly for a more general class of spectra referred to as the higher-moment spectral model, and the variation of $2A_{ii}^N$ with the shape of these spectra is determined. This expression is given by (37). Surprisingly large deviations of $2A_{ii}^N$ from Rotta's form are found when $S_{xx}/\langle u_x^2 \rangle \neq S_{zz}/\langle u_z^2 \rangle$. If these spectra are sufficiently different, then the linear term of $2A_{ii}^N$ can even differ in sign from the Rotta term. In such a case, however, the nonlinear anisotropic terms are clearly essential and not to be ignored.

(4). It is concluded that the Rotta model is not valid in the simple form given by $2\mathbf{A}^N = -(\epsilon/e_0)C\mathbf{a}$. There are two aspects of this invalidity.

(*a*) The off-diagonal element $2A_{xz}^N/a_{xz}$ is intrinsically and quantitatively different from the diagonal elements $2A_{ii}^N/a_{ii}$. This is because the latter depend importantly on the large-wavenumber as well as the small-wavenumber part of the spectrum, whereas the former depends mainly on only the small-wavenumber part of the spectrum. Hence, $2A_{ii}^N/a_{ii} \neq 2A_{xz}^N/a_{xz}$, which contradicts Rotta's model.

(*b*) The diagonal elements $2A_{ii}^N$ are not generally proportional to a_{ii} except for special spectra such as the zero-moment model $(S_{xx}/\langle u_x^2 \rangle = S_{yy}/\langle u_y^2 \rangle = S_{zz}/\langle u_z^2 \rangle)$.

(5). The difference between $2A_{ii}^N/a_{ii}$ and $2A_{xz}^N/a_{xz}$ is calculated for the zeroth-moment model, and it is found that $C_{ii}/C_{xz} \simeq 2$. For that model, Rotta's relation can be used if the difference in C_{xz}/C_{ii} is taken into account; i.e. if one takes $2A_{ii}^N = -(\epsilon/e_0)Ca_{ii}$, $2A_{xz}^N = -(\epsilon/e_0)C_{xz}$, with C_{xz}/C given by (34).

(6). Much emphasis is placed on the distinction between the spectral anisotropy $\mathbf{S}^A \equiv \mathbf{S} - \mathbf{S}^I$ and the stress anisotropy $\mathbf{a} \equiv \langle \mathbf{uu} \rangle - v_0^2\mathbf{I}$. It is pointed out that the Rotta model is the first term in an expansion of $2\mathbf{A}^N$ in powers of $\mathbf{a}$, whereas the theory is an expansion of $2\mathbf{A}^N$ in powers of $\mathbf{S}^A$. The validity of the Rotta model hinges on whether or not the integrals of $\mathbf{S}^A$ such as (14) can be approximated as being proportional to $\mathbf{a}$. Such a proportionality is not valid for all spectra $\mathbf{S}^A$, and the deviations from Rotta's model that we calculate (in §4.2) are deviations from that proportionality. A physical interpretation of these deviations is given.

(7). For nearly homogeneous shear flows, the theoretical linearized magnitudes of $2A_{xx}^N$ and $2A_{yy}^N$ given by (37) (with β_{ix} and β_{iy} determined from the data of Kaimal *et al.* (1972)) is small, much smaller than $2A_{zz}^N$ or than what is given by Rotta's model. This implies that nonlinear terms are essential. The need for a nonlinear correction, in even weak-shear flows, is suggested by the experiments of Champagne *et al.* (1970), wherein is found a significant difference between C_{zz} and C_{yy}. Larger nonlinearities are found by Harris *et al.* (1977). These nonlinearities should be explored in the future. A preliminary calculation indicates that an important nonlinearity proportional to $(\partial U_0/\partial z)^2$ comes from A_{ii}^M, the mean-strain-rate contribution to the pressure–strain rate. Such a term would not be difficult to model.

Appendix A. Angular integration

In this appendix we wish to integrate (24) over the directions of $\mathbf{k}_1$ and $\mathbf{k}_2$ (as was done in I for C_{xz}). We integrate the β_{zz} term first. The β_{zx} and β_{zy} terms will be trivial to integrate afterwards.

Calculation of β_{zz}

To integrate (24), it is convenient to divide γ_z into several parts, the first part of which is dominant (the largest part) and the last part of which is relatively small. (Such a division was also made in appendix B of I.) To obtain such a division we use $k_z = k_{1z} + k_{2z}$ and

$$k_1^2 - (\mathbf{k}_1 . \hat{\mathbf{k}}_2)^2 \equiv k_1^2 [1 - (\hat{\mathbf{k}}_1 . \hat{\mathbf{k}}_2)^2] = k_1^2 [(1 + \hat{\mathbf{k}}_1 . \hat{\mathbf{k}}_2)^2 - 2(\hat{\mathbf{k}}_1 . \hat{\mathbf{k}}_2)(1 + \hat{\mathbf{k}}_1 . \hat{\mathbf{k}}_2)], \quad \text{(A 1)}$$

in B^*. We are thus able to write (20*b–d*) for γ_{zz} as

$$\gamma_{zz} \equiv \gamma_{zz}(1) + \gamma_{zz}(2) + \gamma_{zz}(3) + \gamma_{zz}(4), \quad \text{(A 2a)}$$

$$\gamma_{zz}(1) \equiv \frac{[k_1^2 - (\mathbf{k}_1 . \hat{\mathbf{k}}_2)^2]\, k_{2z}^2}{(k_1^2 + k_2^2)^{\frac{1}{2}}\, k^2}, \quad \text{(A 2b)}$$

$$\gamma_{zz}(2) \equiv \frac{k_1^2 - (\mathbf{k}_1 . \hat{\mathbf{k}}_2)^2}{(k_1^2 + k_2^2)^{\frac{1}{2}}\, k^2} \left[k_{1z} k_{2z} - \tfrac{1}{2} k_z \left(\frac{k_{2x}}{k_{1x} k_{1z}} + \frac{k_{2y}}{k_{1y} k_{1z}} \right) k_{1z}^2 \right], \quad \text{(A 2c)}$$

$$\gamma_{zz}(3) \equiv -\frac{2 k_z k_{2z} k_1^2 (1 + \hat{\mathbf{k}}_1 . \hat{\mathbf{k}}_2)^2}{(k_1^2 + k_2^2)^{\frac{1}{2}}\, k^4} \left[k_{2z}^2 + \left(\frac{k_{2x} k_{2y}}{k_{1x} k_{1y}} - \frac{k_{2x} k_{2z}}{k_{1x} k_{1z}} - \frac{k_{2y} k_{2z}}{k_{1y} k_{1z}} \right) k_{1z}^2 \right], \quad \text{(A 2d)}$$

$$\gamma_{zz}(4) \equiv \left\{ k_z^2 (\hat{\mathbf{k}}_1 . \hat{\mathbf{k}}_2) \left[\frac{4 k_1^2 (1 + \hat{\mathbf{k}}_1 . \hat{\mathbf{k}}_2)}{k^2} - \frac{k_1}{k_2} \right] + k_z k_{1z} \left[1 + \frac{\mathbf{k}_1 . \mathbf{k}_2}{k_2^2} - \frac{2 k_1^2 (1 + \hat{\mathbf{k}}_1 . \hat{\mathbf{k}}_2)^2}{k^2} \right] \right\}$$
$$\times \left[k_{2z}^2 + \left(\frac{k_{2x} k_{2y}}{k_{1x} k_{1y}} - \frac{k_{2x} k_{2z}}{k_{1x} k_{1z}} - \frac{k_{2y} k_{2z}}{k_{1y} k_{1z}} \right) k_{1z}^2 \right] (k_1^2 + k_2^2)^{-\frac{1}{2}} k^{-2}. \quad \text{(A 2e)}$$

Substitution of (A 2) in (24) thus yields the 4 parts of C_z^*:

$$\left. \begin{aligned} \beta_{zz} &\equiv \beta_{zz}^{(1)} + \beta_{zz}^{(2)} + \beta_{zz}^{(3)} + \beta_{zz}^{(4)}, \\ \beta_{zz}^{(j)} &\equiv 6 (\tfrac{1}{2}\pi)^{\frac{1}{2}} \frac{v_0 e_0}{\epsilon} \int \frac{d\mathbf{k}_1}{(2\pi)^3} \int \frac{d\mathbf{k}_2}{4\pi} \frac{\gamma_{zz}(j)\, E(k_2)\, S_{zz}(\mathbf{k}_1)}{k_2^2 e_0 \langle u_z^2 \rangle} \quad (j = 1, \ldots, 4). \end{aligned} \right\} \quad \text{(A 3)}$$

The $\beta_{zz}^{(1)}$ part is largest and simplest. It is given by substituting (A 2) in (A 3) as

$$\beta_{zz}^{(1)} \equiv 6 (\tfrac{1}{2}\pi)^{\frac{1}{2}} \frac{v_0 e_0}{\epsilon} \int \frac{d\mathbf{k}_1}{(2\pi)^3} \int \frac{d\mathbf{k}_2}{4\pi} \frac{k_{2z}^2 [k_1^2 - (\mathbf{k}_1 . \hat{\mathbf{k}}_2)]^2 E(k_2)\, S_{zz}(\mathbf{k}_1)}{k_2^2 e_0 \langle u_z^2 \rangle (k_1^2 + k_2^2)^{\frac{1}{2}} k^2}. \quad \text{(A 4)}$$

Let θ denote the angle between $\mathbf{k}_1$ and $\mathbf{k}_2$:

$$\mathbf{k}_1 . \mathbf{k}_2 = k_1 k_2 \cos\theta.$$

The dependence of the integrand of (A 4) on θ is given, with $k^2 \equiv |\mathbf{k}_1 + \mathbf{k}_2|^2$, by

$$\frac{k_{2z}^2 [k_1^2 - (\mathbf{k}_1 . \hat{\mathbf{k}}_2)^2]}{k_1^2 + k_2^2 + 2 \mathbf{k}_1 . \mathbf{k}_2} = \frac{k_{2z}^2 k_1^2 (1 - \cos^2\theta)}{k_1^2 + k_2^2 + 2 k_1 k_2 \cos\theta}. \quad \text{(A 5)}$$

As in I, the chief assumption we shall make to evaluate $\beta_{zz}^{(1)}$, as well as the other $\beta_{zz}^{(j)}$ is that the main contribution to the (scalar) k_1 and k_2 integrations in (A 4) comes from $k_1 \simeq k_2$. This assumption was found in I (§ 7*d*) to cause an error of only about 2% and greatly simplifies the integrations in (A 4). One basis of this assumption is that the factor $k_{2z}^2 k_1^2/(k_1^2+k_2^2+2k_1k_2\cos\theta)$ is itself a maximum when $k_1 \simeq k_2$ and decreases fairly rapidly when k_1/k_2 varies away from unity. The validity of the assumption $k_1 \simeq k_2$ is enhanced for the zero-moment spectral model used in §4.1 because S_{ii} and E have their maximum values at the same wavenumber k_L. That assumption is weaker for the more complex spectral model of §4.3 (that model is discussed separately in appendix C). Here, then, we approximate (A 5) by

$$\frac{k_{2z}^2[k_1^2-(\mathbf{k}_1.\hat{\mathbf{k}}_2)^2]}{k_1^2+k_2^2+2\mathbf{k}_1.\mathbf{k}_2} \simeq \frac{k_{2z}^2 k_1^2(1-\cos^2\theta)}{(k_1^2+k_2^2)(1+\cos\theta)} = \frac{k_{2z}^2 k_1^2(1-\cos\theta)}{k_1^2+k_2^2}. \qquad \text{(A 6)}$$

When (A 6) is substituted into the integrand of (A 4), the $\cos\theta$ vanishes because the remainder of the integrand is independent of θ. Therefore, substitution of (A 6) in (A 4) yields

$$\beta_{zz}^{(1)} = 6(\tfrac{1}{2}\pi)^{\frac{1}{2}}\frac{v_0 e_0}{\epsilon}\int\frac{d\mathbf{k}_1}{(2\pi)^3}\int\frac{d\mathbf{k}_2}{4\pi}\frac{k_{2z}^2 k_1^2 E(k_2)\,S_{zz}(\mathbf{k}_1)}{(k_1^2+k_2^2)^{\frac{3}{2}}\,k_2^2 e_0\langle u_z^2\rangle}. \qquad \text{(A 7)}$$

We next express the $\mathbf{k}_1$ and $\mathbf{k}_2$ integrals in spherical co-ordinates, e.g.

$$\int d\mathbf{k}_2 \equiv \int_0^\infty k_2^2\,dk_2\int_0^\pi d\theta_2\sin\theta_2\int_0^{2\pi}d\phi_2,$$

where θ_2 is the angle $\mathbf{k}_2$ makes with the $\hat{\mathbf{x}}$-axis ($k_{2x} = k_2\cos\theta_2$), and ϕ_2 is the (azimuthal) angle of $\mathbf{k}_2$ in the plane perpendicular to $\hat{\mathbf{x}}$. We then perform the θ_2 and ϕ_2 integrations in (A 7) as follows:

$$\frac{1}{4\pi}\int_0^{2\pi}d\phi_2\int_0^\pi d\theta_2\sin\theta_2\,k_{2x}^2 = \tfrac{1}{3}k_2^2. \qquad \text{(A 8)}$$

We also integrate $S_{zz}(\mathbf{k}_1)$ over a spherical shell of radius k_1 to obtain a scalar spectrum, which we denote by $E_{zz}(k_1)$:

$$E_{ii}(k_1) \equiv \frac{k_1^2}{(2\pi)^3}\int_0^{2\pi}d\phi_1\int_0^\pi d\theta_1\sin\theta_1\,S_{ii}(\mathbf{k}_1) \quad (i = x, y, z), \qquad \text{(A 9)}$$

where θ_1 is the angle $\mathbf{k}_1$ makes with the $\mathbf{x}$-axis and ϕ_1 is the azimuthal angle of $\mathbf{k}_1$. Substitution of (A 8) and (A 9) in (A 7) finally gives the scalar integral

$$\beta_{zz}^{(1)} = 2(\tfrac{1}{2}\pi)^{\frac{1}{2}}\frac{v_0 e_0}{\epsilon}\int_0^\infty dk_1\int_0^\infty dk_2\,\frac{k_2^2 E(k_2)\,E_{zz}(k_1)\,k_1^2}{(k_1^2+k_2^2)^{\frac{3}{2}}\,e_0\langle u_z^2\rangle}. \qquad \text{(A 10)}$$

Next, we evaluate $\beta_{zz}^{(2)}$. The expression for $\beta_{zz}^{(2)}$ is given by substitution of $\gamma_{zz}(2)$ from (A 2), in (A 3):

$$\beta_{zz}^{(2)} = 6(\tfrac{1}{2}\pi)^{\frac{1}{2}}\frac{v_0 e_0}{\epsilon}\int\frac{d\mathbf{k}_1}{(2\pi)^3}\int\frac{d\mathbf{k}_2}{4\pi}\left[k_{1z}k_{2z}-\tfrac{1}{2}k_z\left(\frac{k_{2x}}{k_{1x}k_{1z}}+\frac{k_{2y}}{k_{1y}k_{1z}}\right)k_{1z}^2\right] \times\frac{[k_1^2-(\mathbf{k}_1.\hat{\mathbf{k}}_2)]^2\,E(k_2)\,S_{zz}(\mathbf{k}_1)}{(k_1^2+k_2^2)^{\frac{1}{2}}\,k^2k_2^2\,e_0\langle u_z^2\rangle}. \qquad \text{(A 11)}$$

This integrates very easily if we use approximation (A 6):

$$\frac{k_1^2-(\mathbf{k}_1.\hat{\mathbf{k}}_2)^2}{k^2} \simeq \frac{k_1^2(1-\cos\theta)}{k_1^2+k_2^2} = \frac{k_1^2}{k_1^2+k_2^2}\left[1-\frac{k_{1x}k_{2x}+k_{1y}k_{2y}+k_{1z}k_{2z}}{k_1k_2}\right], \quad \text{(A 12)}$$

where we have used the identity $\cos\theta \equiv (\mathbf{k}_1.\mathbf{k}_2)/k_1k_2$. We substitute (A 12) into (A 11) and note that all terms odd in k_{2x}, k_{2y} or k_{2z} vanish because $E(k_2)$ is an even function of of these components. Hence (A 12) reduces to

$$\beta_{zz}^{(2)} = 6(\tfrac{1}{2}\pi)^{\frac{1}{2}}\frac{v_0e_0}{\epsilon}\int\frac{d\mathbf{k}_1}{(2\pi)^3}\int\frac{d\mathbf{k}_2k_1k_{1z}^2[k_{2z}^2-\frac{1}{2}(k_{1x}^2+k_{1y}^2)]E(k_2)S_{zz}(\mathbf{k}_1)}{4\pi(k_1^2+k_2^2)^{\frac{3}{2}}k_2^3e_0\langle u_z^2\rangle} = 0. \quad \text{(A 13)}$$

The right-hand side of (A 13) vanishes because $E(k_2)$ is a scalar function of k_2 so that the integral of $k_{2z}^2-\frac{1}{2}(k_{2x}^2+k_{2y}^2)$ over $\mathbf{k}_2$ vanishes; i.e. $\int d\mathbf{k}_2k_{2z}^2/k_2^2 = \int d\mathbf{k}_2k_{2y}^2/k_2^2 = \int d\mathbf{k}_2k_{2x}^2/k_2^2$.

The expression for $\beta_{zz}^{(3)}$ is given by substitution of $\gamma_{zz}(3)$, from (A 2), into (A 3). The form of $\gamma_{zz}(3)$ is greatly simplified by using

$$\frac{(1+\hat{\mathbf{k}}_1.\hat{\mathbf{k}}_2)^2}{(k_1^2+k_2^2+2\hat{\mathbf{k}}_1.\hat{\mathbf{k}}_2)^2} \simeq \frac{(1+\hat{\mathbf{k}}_1.\hat{\mathbf{k}}_2)^2}{(k_1^2+k_2^2)^2(1+\hat{\mathbf{k}}_1.\hat{\mathbf{k}}_2)^2} = \frac{1}{(k_1^2+k_2^2)^2}, \quad \text{(A 14)}$$

where we have again used approximation (A 6). Substitution of $\gamma_{zz}(3)$ and (A 14) in (A 3) yields $\beta_{zz}^{(3)}$ as

$$\beta_{zz}^{(3)} = -6(\tfrac{1}{2}\pi)^{\frac{1}{2}}\frac{v_0e_0}{\epsilon}\int\frac{d\mathbf{k}_1}{(2\pi)^3}\int\frac{d\mathbf{k}_2}{4\pi}\left[k_{2z}^2+\left(\frac{k_{2x}k_{2y}}{k_{1x}k_{1y}}-\frac{k_{2x}k_{2z}}{k_{1x}k_{1y}}-\frac{k_{2y}k_{2z}}{k_{1y}k_{1z}}\right)k_{1z}^2\right] \times\frac{2k_zk_{2z}k_1^2E(k_2)S_{zz}(\mathbf{k}_1)}{(k_1^2+k_2^2)^{\frac{5}{2}}k_2^2e_0\langle u_z^2\rangle}. \quad \text{(A 15)}$$

Those terms in (A 14) which are odd in k_{2x}, k_{2y} or k_{2z} vanish, so that using $k_z = k_{1z}+k_{2z}$, (A 15) reduces to

$$\beta_{zz}^{(3)} = -6(\tfrac{1}{2}\pi)^{\frac{1}{2}}\frac{v_0e_0}{\epsilon}\int\frac{d\mathbf{k}_1}{(2\pi)^3}\int\frac{d\mathbf{k}_2}{4\pi}\frac{2k_1^2k_{2z}^4E(k_2)S_{zz}(\mathbf{k}_1)}{(k_1^2+k_2^2)^{\frac{5}{2}}k_2^2e_0\langle u_z^2\rangle}. \quad \text{(A 16)}$$

The $\mathbf{k}_1$ and $\mathbf{k}_2$ integrals are next expressed in spherical co-ordinates as was done for $\beta_{zz}^{(1)}$. The θ_2 and ϕ_2 integrations in (A 16) are given by

$$\frac{1}{4\pi}\int_0^{2\pi}d\phi_2\int_0^{\pi}d\theta_2\sin\theta_2k_{2z}^4 = \tfrac{1}{5}k_2^4, \quad \text{(A 17)}$$

and the θ_1 and ϕ_1 integrations are the same as (A 9). Substitution of (A 9) and (A 17) in (A 16) gives $\beta_{zz}^{(3)}$ as the scalar integral

$$\beta_{zz}^{(3)} = -\frac{12}{5}(\tfrac{1}{2}\pi)^{\frac{1}{2}}\int_0^{\infty}dk_1\int_0^{\infty}dk_2\frac{k_1^2k_2^4E(k_2)E_{zz}(k_1)}{(k_1^2+k_2^2)^{\frac{5}{2}}e_0\langle u_z^2\rangle}. \quad \text{(A 18)}$$

This expression is similar in form to that given by (A 10) for $\beta_{zz}^{(1)}$. The integrand of (A 18) contains the additional factor $k_2^2(k_1^2+k_2^2)^{-1}$. However, as pointed out for (A 4), the main contribution to the integrations in (A 18) come from $k_1 \simeq k_2$. (This assumption is not used for the more general calculation of $\beta_{zz}^{(3)}$ in appendix C.) In the present case we can take $k_2^2(k_1^2+k_2^2)^{-1} \simeq \frac{1}{2}$ in (A 18), and, comparing with (A 10), obtain

$$\beta_{zz}^{(3)} \simeq -\tfrac{3}{5}\beta_{zz}^{(1)}. \quad \text{(A 18')}$$

The expression for $\beta_{zz}^{(4)}$ is given by substitution of $\gamma_{zz}(4)$ in (A 3). It is then seen that $\beta_{zz}^{(4)}$ is more complex than $\beta_{zz}^{(1)}$, $\beta_{zz}^{(2)}$ and $\beta_{zz}^{(2)}$ because it contains more terms, but mainly because it contains the factor

$$k^{-2} = (k_1^2+k_2^2+2\mathbf{k}_1.\mathbf{k}_2)^{-1} \simeq (k_1^2+k_2^2)^{-1}(1+\hat{\mathbf{k}}_1.\hat{\mathbf{k}}_2)^{-1}$$

(the term $(1+\hat{\mathbf{k}}_1.\hat{\mathbf{k}}_2)^{-1}$ cancels out in the other β_{zz}^{j}). The quantity $(1+\hat{\mathbf{k}}_1.\hat{\mathbf{k}}_2)^{-1}$ can be integrated by the straightforward expansion

$$(1+\hat{\mathbf{k}}_1.\hat{\mathbf{k}}_2)^{-1} = 1-\hat{\mathbf{k}}_1.\hat{\mathbf{k}}_2+(\hat{\mathbf{k}}_1.\hat{\mathbf{k}}_2)^2+\ldots,$$

and, fortunately, the higher-order terms sum to a very small value (as we have found). The other approximation we use in $\beta_{zz}^{(4)}$ is the following substitution of (A 6) in a factor of $\beta_{zz}^{(4)}$:

$$\frac{4k_1^2(1+\hat{\mathbf{k}}_1.\hat{\mathbf{k}}_2)}{k^2}-\frac{k_1}{k_2} \simeq \frac{4k_1^2}{k_1^2+k_2^2}-\frac{k_1}{k_2}.$$

We will not present the remainder of the calculation of $\beta_{zz}^{(4)}$ here because it is lengthy and because $\beta_{zz}^{(4)}$ is small. We only quote the result as follows:

$$\beta_{zz}^{(4)} \simeq -6(\tfrac{1}{2}\pi)^{\frac{1}{2}}\frac{v_0 e_0}{\epsilon}\int_0^\infty dk_1\int_0^\infty dk_2\,\frac{k_2^2 E(k_2)\,E_{zz}(k_1)\,k_1^2}{(k_1^2+k_2^2)^{\frac{3}{2}}\,e_0\langle u_z^2\rangle} \times\left[0{\cdot}013\left(\frac{4k_1}{k_1^2+k_2^2}-\frac{k_1}{k_2}\right)\frac{k_2}{k_1}-\frac{1}{9}\left(1-\frac{2k_1^2}{k_1^2+k_2^2}\right)\right]. \quad \text{(A 19)}$$

As with (A 18), if we use the fact that the main contribution to the integral in (A 19) comes from $k_1 \simeq k_2$, then (A 19) becomes simply

$$\beta_{zz}^{(4)} \simeq -0{\cdot}039\beta_{zz}^{(1)}. \quad \text{(A 19}')$$

Finally, β_{zz} is obtained by substituting (A 10), (A 13), (A 18′) and (A 19′) for $\beta_{zz}^{(j)}$ into (A 3)

$$\beta_{zz} = 0{\cdot}72(\tfrac{1}{2}\pi)^{\frac{1}{2}}\frac{v_0 e_0}{\epsilon}\int_0^\infty dk_1\int_0^\infty dk_2\,\frac{k_2^2 k_1^2 E(k_2)\,E_{zz}(k_1)}{(k_1^2+k_2^{2\frac{3}{2}})\,e_0\langle u_z^2\rangle}. \quad \text{(A 20)}$$

The overall accuracy of (A 20) is dependent on the approximations using $k_1 \simeq k_2$ such as (A 6). These approximations are discussed in appendix C.

Calculation of β_{zx} and β_{zy}

The integration of (24) for β_{zx} and β_{zy} is very similar to the integration of β_{zz} just given. Furthermore, it is seen that β_{zy} is converted into β_{zx} by exchanging k_{1x} with k_{1y} and k_{2x} with k_{2y}. For this reason, it will only be necessary to calculate β_{zx}.

To perform the $\mathbf{k}_1$ and $\mathbf{k}_2$ integrations of β_{zx} it is convenient to divide γ_{zx} into 3 parts as follows:

$$\gamma_{zx} \equiv \gamma_{zx}(1)+\gamma_{zx}(2)+\gamma_{zx}(3), \quad \text{(A 21}a)$$

$$\gamma_{zx}(1) \equiv \frac{k_1^2-(\mathbf{k}_1.\hat{\mathbf{k}}_2)^2}{(k_1^2+k_2^2)^{\frac{1}{2}}\,k^2}\left[\tfrac{1}{2}k_z\left(\frac{k_{2y}}{k_{1y}k_{1z}}-\frac{k_{2x}}{k_{1x}k_{1z}}\right)k_{1z}^2\right], \quad \text{(A 21}b)$$

$$\gamma_{zx}(2) \equiv -\frac{2k_z k_{2z} k_1^2(1+\hat{\mathbf{k}}_1.\hat{\mathbf{k}}_2)^2}{(k^2+k_2^2)^{\frac{1}{2}}\,k^4}\left[k_{2x}^2+\left(\frac{k_{2y}k_{2z}}{k_{1y}k_{1z}}-\frac{k_{2x}k_{2y}}{k_{1x}k_{1y}}-\frac{k_{2x}k_{2z}}{k_{1x}k_{1z}}\right)k_{1x}^2\right], \quad \text{(A 21}c)$$

$$\gamma_{zx}(3) \equiv \left\{k_z^2(\hat{\mathbf{k}}_1.\hat{\mathbf{k}}_2)\left[\frac{4k_1^2(1+\hat{\mathbf{k}}_1.\hat{\mathbf{k}}_2)}{k^2}-\frac{k_1}{k_2}\right]+k_z k_{1z}\left[1+\frac{\mathbf{k}_1.\mathbf{k}_2}{k_2^2}-\frac{2k_1^2(1+\hat{\mathbf{k}}_1.\hat{\mathbf{k}}_2)^2}{k^2}\right]\right\}$$
$$\times\left[k_{2x}^2+\left(\frac{k_{2y}k_{2z}}{k_{1y}k_{1z}}-\frac{k_{2x}k_{2y}}{k_{1x}k_{1y}}-\frac{k_{2x}k_{2z}}{k_{1x}k_{1z}}\right)k_{1x}^2\right](k_1^2+k_2^2)^{-\frac{1}{2}}k^{-2}. \quad (\text{A } 21d)$$

(It is seen that $\gamma_{zx}(1)$, $\gamma_{zx}(2)$, $\gamma_{zx}(3)$ are analogous in form to $\gamma_{zz}(2)$, $\gamma_{zz}(3)$, $\gamma_{zz}(4)$ respectively; γ_{zx} has no term analogous to $\gamma_{zz}(1)$.) Substitution of (A 21) in (24) yields β_{zx} divided into 3 parts:

$$\left.\begin{aligned}\beta_{zx} &\equiv \beta_{zx}^{(1)}+\beta_{zx}^{(2)}+\beta_{zx}^{(3)}\\ \beta_{zx}^{(j)} &\equiv 6(\tfrac{1}{2}\pi)^{\frac{1}{2}}\frac{v_0e_0}{\epsilon}\int\frac{d\mathbf{k}_1}{(2\pi)^3}\int\frac{d\mathbf{k}_2}{4\pi}\frac{\gamma_{zx}(j)\,E(k_2)\,S_{xx}(\mathbf{k}_1)}{k_2^2e_0\langle u_x^2\rangle}.\end{aligned}\right\} \quad (\text{A } 22)$$

Each of these terms is calculated in the same way as was done for its counterpart $\beta_{zz}^{(j)}$ above.

Thus, $\beta_{zx}^{(1)}$ is given by

$$\beta_{zx}^{(1)} = 6(\tfrac{1}{2}\pi)^{\frac{1}{2}}\frac{v_0e_0}{\epsilon}\int\frac{d\mathbf{k}_1}{(2\pi)^3}\int\frac{d\mathbf{k}_2}{4\pi}\left[\tfrac{1}{2}k_z\left(\frac{k_{2y}}{k_{1y}k_{1z}}-\frac{k_{2x}}{k_{1x}k_{1z}}\right)k_{1x}^2\right.$$
$$\left.\times\frac{[k_1^2-(\mathbf{k}_1.\hat{\mathbf{k}}_2)^2]\,E(k_2)\,S_{xx}(\mathbf{k}_1)}{(k_1^2+k_2^2)^{\frac{1}{2}}k^2k_2^2e_0\langle u_z^2\rangle}\right]. \quad (\text{A } 22')$$

We next substitute (A 12) into (A 22), multiply out the various components of $\mathbf{k}_1$ and $\mathbf{k}_2$, and use the fact that terms that are odd in k_{2x}, k_{2y} or k_{2z} vanish when integrated over $\mathbf{k}_2$. We thus obtain

$$\beta_{zx}^{(1)} = 6(\tfrac{1}{2}\pi)^{\frac{1}{2}}\frac{v_0e_0}{\epsilon}\int\frac{d\mathbf{k}_1}{(2\pi)^3}\int\frac{d\mathbf{k}_2}{4\pi}\frac{\frac{1}{2}(k_{2y}^2-k_{2x}^2)\,k_{1x}^2k_1E(k_2)\,S_{xx}(\mathbf{k}_1)}{(k_1^2+k_2^2)^{\frac{3}{2}}k_2^3e_0\langle u_x^2\rangle} = 0. \quad (\text{A } 23)$$

$\beta_{zx}^{(1)}$ in (A 23) is equal to zero because the angular integration of k_{2x}^2 over the directions of $\mathbf{k}_2$ is equal to the angular integration of k_{2y}^2.

The evaluation of $\beta_{zx}^{(2)}$ is very similar to that of $\beta_{zz}^{(3)}$ above. The expression for $\beta_{zx}^{(2)}$ is obtained by substituting (A 21) into (A 22), and is substantially simplified by using (A 14). We thus find that $\beta_{zx}^{(2)}$ is given by

$$\beta_{zx}^{(2)} = -6(\tfrac{1}{2}\pi)^{\frac{1}{2}}\frac{v_0e_0}{\epsilon}\int\frac{d\mathbf{k}_1}{(2\pi)^3}\int\frac{d\mathbf{k}_2}{4\pi}\left[k_{2x}^2+\left(\frac{k_{2y}k_{2z}}{k_{1y}k_{1z}}-\frac{k_{2x}k_{2y}}{k_{1x}k_{1y}}-\frac{k_{2x}k_{2z}}{k_{1x}k_{1z}}\right)k_{1x}^2\right]$$
$$\times\frac{2k_zk_{2z}k_1^2E(k_2)\,S_{xx}(\mathbf{k}_1)}{(k_1^2+k_2^2)^{\frac{3}{2}}k_2^2e_0\langle u_x^2\rangle}, \quad (\text{A } 24a)$$

$$\beta_{zx}^{(2)} = -6(\tfrac{1}{2}\pi)^{\frac{1}{2}}\frac{v_0e_0}{\epsilon}\int\frac{d\mathbf{k}_1}{(2\pi)^3}\int\frac{d\mathbf{k}_2}{4\pi}\frac{2k_{2x}^2k_{2z}^2k_1^2E(k_2)\,S_{xx}(\mathbf{k}_1)}{(k_1^2+k_2^2)^{\frac{3}{2}}k_2^2e_0\langle u_x^2\rangle}. \quad (\text{A } 24b)$$

The last step in (A 24) follows from the fact that terms odd in k_{2x}, k_{2y} or k_{2z} vanish. The integration over the spherical angles θ_2 and ϕ_2 given by

$$\frac{1}{4\pi}\int_0^{2\pi}d\phi_2\int_0^{\pi}d\theta_2\sin\theta_2\,k_{2x}^2k_{2z}^2 = \tfrac{1}{15}k_2^4, \quad (\text{A } 25)$$

and the θ_1 and ϕ_1 integrations are the same as in (A 9). Substitution of (A 9) and (A 25) in (A 24) gives $\beta_{zx}^{(2)}$ as

$$\beta_{zx}^{(2)} = -\tfrac{4}{5}(\tfrac{1}{2}\pi)^{\frac{1}{2}}\frac{v_0 e_0}{\epsilon}\int_0^\infty dk_1 \int_0^\infty dk_2 \frac{k_1^2 k_2^4 E(k_2)\, E_{xx}(k_1)}{(k_1^2+k_2^2)^{\frac{5}{2}}\, e_0\langle u_x^2\rangle}. \tag{A 26}$$

This expression can be related to $\beta_{zz}^{(1)}$ in (A 10) by using our basic assumption (that the main contribution to the integration comes from $k_1 \simeq k_2$) so as to approximate $k_2^2(k_1^2+k_2^2)^{-1} = \frac{1}{2}$ in (A 26). (This approximation can be shown to be exact in (A 26) for the zero-moment model of §4.1 in which $E(k)/e_0 = E_{xx}/\langle u_x^2\rangle$. The approximation is not used for the more complex model of §4.3.) We thus have

$$\beta_{zx}^{(2)} = -\tfrac{2}{5}(\tfrac{1}{2}\pi)^{\frac{1}{2}}\frac{v_0 e_0}{\epsilon}\int_0^\infty dk_1 \int_0^\infty dk_2 \frac{k_1^2 k_2^2 E(k_2)\, E_{xx}(k_1)}{(k_1^2+k_2^2)^{\frac{3}{2}}\, e_0\langle u_x^2\rangle}, \tag{A 26'}$$

which has the same form as $\beta_{zz}^{(1)}$ with $E_{zz}/\langle u_z^2\rangle$ replaced by $E_{xx}/\langle u_x^2\rangle$.

The quantity $\beta_{zx}^{(3)}$ is relatively very small, and is also more complex than $\beta_{zx}^{(2)}$. It is calculated in the same way as was done for $\beta_{zz}^{(4)}$. We merely present the result of this calculation as follows:

$$\beta_{zx}^{(3)} \simeq 6(\tfrac{1}{2}\pi)^{\frac{1}{2}}\frac{v_0 e_0}{\epsilon}\int_0^\infty dk_1 \int_0^\infty dk_2 \frac{k_1^2 k_2^2 E(k_2)\, E_{xx}(k_1)}{(k_1^2+k_2^2)^{\frac{3}{2}}\, e_0\langle u_x^2\rangle} \cdot \times\left[0{\cdot}0064\left(\frac{4k_1^2}{k_1^2+k_2^2}-\frac{k_1}{k_2}\right)\frac{k_2}{k_1}+\frac{1}{9}\left(1-\frac{2k_1^2}{k_1^2+k_2^2}\right)\right]. \tag{A 27}$$

As with (A 26) and (A 19) the main contribution to the integral comes from $k_1 \simeq k_2$, so that (A 27) reduces to

$$\beta_{zx}^{(3)} \simeq -0{\cdot}096\beta_{zx}^{(2)}. \tag{A 27'}$$

Finally, β_{zx} is obtained by substituting (A 23), (A 26') and (A 27') into (A 22):

$$\beta_{zx} = -0{\cdot}36(\tfrac{1}{2}\pi)^{\frac{1}{2}}\frac{v_0 e_0}{\epsilon}\int dk_1 \int dk_2 \frac{k_1^2 k_2^2 E(k_2)\, E_{xx}(k_1)}{(k_1^2+k_2^2)^{\frac{1}{2}}\, e_0\langle u_x^2\rangle}. \tag{A 28}$$

Note that β_{zx} equals $-\frac{1}{2}\beta_{zz}$ when $E_{xx}/\langle u_x^2\rangle = E_{zz}/\langle u_z^2\rangle$.

The expression for β_{zy} follows from (A 28) by a simple symmetry consideration. That is, it is seen in (20) that γ_{zy} is obtained from γ_{zx} by interchanging x and y (i.e. interchanging k_{1x} and k_{1y}, k_{2x} and k_{2y}, S_{xx} and S_{yy}, and u_x and u_y). Hence, β_{zy} is obtained from β_{zx} by simply interchanging x and y:

$$\beta_{zy} = -0{\cdot}36(\tfrac{1}{2}\pi)^{\frac{1}{2}}\frac{v_0 e_0}{\epsilon}\int dk_1 \int dk_2 \frac{k_1^2 k_2^2 E(k_2)\, E_{yy}(k_1)}{(k_1^2+k_2^2)^{\frac{1}{2}}\, e_0\langle u_y^2\rangle}. \tag{A 29}$$

Appendix B. Derivation of $\mathbf{A}_{xx}^N$ and $\mathbf{A}_{yy}^N$

The purpose of this appendix is to derive expressions for $2A_{xx}^N$ and $2A_{yy}^N$ analogous to the expressions derived for $2A_{zz}^N$; expressions such as (22) and (27). To do this, we write the diagonal pressure–strain-rate elements in Fourier components as was done for (2),

$$\langle p\,\partial u_n/\partial n\rangle = -\frac{1}{(2\pi)^3 V}\int d\mathbf{k}\langle u_n^*(\mathbf{k},t)\, ik_n p(\mathbf{k},t)\rangle \quad (n = x,y,z), \tag{B 1}$$

and then we substitute the p expression (5) into (B 1) to obtain

$$\left.\begin{aligned} 2\rho_0^{-1}\langle p\,\partial u_n/\partial n\rangle &= 2A_{nn}^{N} + 2A_{nn}^{M},\\ A_{nn}^{N} &\equiv -\frac{i}{(2\pi)^3 V}\int d\mathbf{k}\,k_n\langle u_n^*(\mathbf{k},t)\,N(\mathbf{k},t)\rangle,\\ A_{nn}^{M} &\equiv \frac{2}{(2\pi)^3 V}\int d\mathbf{k}\,\frac{k_x k_n}{k^2}\langle u_z^*(\mathbf{k},t)\,u_n(\mathbf{k},t)\rangle\frac{\partial U_0}{\partial z}.\end{aligned}\right\} \quad \text{(B 2)}$$

It can be seen from (B 2) that the expressions for A_{zz}^N in §§3 and 4 can be changed to expressions for A_{xx}^N by suitably interchanging the subscripts z and x. Similarly, A_{yy}^N can be obtained by suitably interchanging z and y. Hence, from (22) and (24) we can write A_{xx}^N and A_{yy}^N as

$$2A_{nn}^N = -\frac{\epsilon}{e_0}[\beta_{nx}\langle u_x^2\rangle + \beta_{ny}\langle u_y^2\rangle + \beta_{nz}\langle u_z^2\rangle] \quad (n = x,y,z), \tag{B 3}$$

$$\beta_{ni} \equiv 6(\tfrac{1}{2}\pi)^{\frac{1}{2}}\frac{v_0 e_0}{\epsilon}\int\frac{d\mathbf{k}_1}{(2\pi)^3}\int\frac{d\mathbf{k}_2}{4\pi}\frac{\gamma_{ni}E(k_2)\,S_{ii}(\mathbf{k}_1)}{k_2^2 e_0\langle u_i^2\rangle} \quad (n,i = x,y,z), \tag{B 4}$$

where γ_{ni} is determined by symmetry from (20) to be

$$\begin{aligned}\gamma_{xx} \equiv \frac{k_1^2-(\mathbf{k}_1.\hat{\mathbf{k}}_2)^2}{(k_1^2+k_2^2)^{\frac{1}{2}}k^2}&\left[k_x k_{2x} - \tfrac{1}{2}k_x\left(\frac{k_{2z}}{k_{1x}k_{1z}}+\frac{k_{2y}}{k_{1x}k_{1y}}\right)k_{1y}^2\right]\\ &+ B_x^*\left[k_{2x}^2 + \left(\frac{k_{2y}k_{2z}}{k_{1y}k_{1z}} - \frac{k_{2x}k_{2z}}{k_{1x}k_{1z}} - \frac{k_{2x}k_{2y}}{k_{1x}k_{1y}}\right)k_{1x}^2\right],\end{aligned} \tag{B 5a}$$

$$\begin{aligned}\gamma_{yy} \equiv \frac{k_1^2-(\mathbf{k}_1.\hat{\mathbf{k}}_2)^2}{(k_1^2+k_2^2)^{\frac{1}{2}}k^2}&\left[k_y k_{2y} - \tfrac{1}{2}k_y\left(\frac{k_{2x}}{k_{1x}k_{1y}}+\frac{k_{2z}}{k_{1y}k_{1z}}\right)k_{1y}^2\right]\\ &+ B_y^*\left[k_{2y}^2 + \left(\frac{k_{2x}k_{2z}}{k_{1x}k_{1z}} - \frac{k_{2x}k_{2y}}{k_{1x}k_{1y}} - \frac{k_{2y}k_{2z}}{k_{1y}k_{1z}}\right)k_{1y}^2\right],\end{aligned} \tag{B 5b}$$

$$\begin{aligned}\gamma_{ni} \equiv \frac{k_1^2-(k_1.\hat{\mathbf{k}}_2)^2}{(k_1^2+k_2^2)^{\frac{1}{2}}k^2}&\left[\tfrac{1}{2}k_n\left(\frac{k_{2s}}{k_{1s}k_{1n}}-\frac{k_{2i}}{k_{1i}k_{1n}}\right)k_{1i}^2\right]\\ &+ B_n^*\left[k_{2i}^2 + \left(\frac{k_{2s}k_{2n}}{k_{1s}k_{1n}} - \frac{k_{2s}k_{2i}}{k_{1s}k_{1i}} - \frac{k_{2n}k_{2i}}{k_{1n}k_{1i}}\right)k_{1i}^2\right] \quad (n \neq i \neq s),\end{aligned} \tag{B 5c}$$

$$B_n^* \equiv \left\{k_n k_{1n} - k_n k_{2n}\frac{\mathbf{k}_1.\mathbf{k}_2}{k_2^2} - 2\frac{k_n^2}{k^2}[k_1^2 - (\mathbf{k}_1.\hat{\mathbf{k}}_2)^2]\right\}(k_1^2+k_2^2)^{-\frac{1}{2}}k^{-2}. \tag{B 5d}$$

We thus have (B 3) as the generalization of (22) for $2A_{nn}^N$ (where $n = x,y,z$). The required coefficients γ_{xx}, γ_{yy} and γ_{ni} are given by (B 5). Note that (B 3) agrees with Rotta's model if $\beta_{xy} = \beta_{xz} = \beta_{yz} = -\frac{1}{2}\beta_{nn}$.

We integrate over the spherical angles of $\mathbf{k}_1$ and $\mathbf{k}_2$ in (B 4), exactly as in appendix A. The result is the same as (25) (except for interchanged subscripts):

$$\left.\begin{aligned}\beta_{ni} &= d_{ni}(\tfrac{1}{2}\pi)^{\frac{1}{2}}\frac{v_0 e_0}{\epsilon}\int_0^\infty dk_1\int_0^\infty dk_2\,\frac{k_1^2+k_2^2E(k_2)\,E_{ii}}{(k_1^2+k_2^2)^{\frac{3}{2}}e_0\langle u_z^2\rangle}\\ d_{nn} &= 0{\cdot}72, \quad d_{ni} = -0{\cdot}36 \quad (n \neq i).\end{aligned}\right\} \quad \text{(B 6)}$$

The generalization of (27) to apply to $2A^N_{xx}$ and $2A^N_{yy}$ can now be obtained immediately by substitution of (B 6) into (B 3). We thus have

$$2A^N_{nn} = -0{\cdot}72(\tfrac{1}{2}\pi)^{\frac{1}{2}}\frac{v_0 e_0}{\epsilon}\int_0^\infty dk_1 \int_0^\infty dk_2 \frac{k_1^2 k_2^2 E(k_2)}{(k_1^2+k_2^2)^{\frac{3}{2}} e_0}[\tfrac{3}{2}E_{nn}(k_1) - E(k_1)], \quad \text{(B 7)}$$

which is seen to agree with (27) when we use $E \equiv \frac{1}{2}(E_{xx}+E_{yy}+E_{zz})$.

Substituting (29), the *zeroth-moment model*, into (B 7) we obtain the Rotta relation

$$\left.\begin{aligned} 2A^N_{nn} &= -\frac{\epsilon}{e_0}C_{nn}[\langle u_n^2\rangle - (\tfrac{2}{3})e_0] \quad (n = x,y,z),\\ C_{nn} &\equiv \tfrac{3}{2}\beta_{nn} \quad \text{(for model (29))}, \end{aligned}\right\} \quad \text{(B 8)}$$

which generalizes (30) to all the diagonal components. In addition, for the zeroth-moment model it is seen, from (B 6), that C_{nn} is the same for all n:

$$C_{xx} = C_{yy} = C_{zz} \quad \text{(for model (29))}. \quad \text{(B 9)}$$

Appendix C. Higher-moment model

The purpose of this appendix is to evaluate numerically the coefficients β_{ij} in (37), and thus determine A^N_{ii}, for a spectral model in which the peak of $E_{xx}(k)$ does not occur at the same wavenumber as does the peak of E_{zz}. Such a model is suggested by the data of Kaimal *et al.* (1972). We refer to this model as the higher-moment model because, as we will see, all the moments of $E_{xx}/\langle u_x^2\rangle$ differ from those of $E_{zz}/\langle u_z^2\rangle$. To specify the model of E_{ii}, we use three considerations suggested by the experiments: (1) E_{xx}, E_{yy} and E_{zz} approach each other for large k (approximate local isotropy at large k); (2) $\langle u_x^2\rangle > \langle u_y^2\rangle > \langle u_z^2\rangle$; and (3) E_{xx} peaks (has its maximum) at a smaller wavenumber than does E_{zz}. The model we chose for E_{ii} is illustrated in figure 1. It is given by

$$\left.\begin{aligned} E_{xx}(k) &= \begin{cases}\alpha\epsilon^{\frac{2}{3}}k^{-\frac{5}{3}} & (k \geqslant k_L),\\ \alpha\epsilon^{\frac{2}{3}}(k_L)^{-m-\frac{5}{3}}k^m & (k \leqslant k_L);\end{cases}\\ E_{yy} = E_{zz} &= \begin{cases}\alpha\epsilon^{\frac{2}{3}}k^{-\frac{5}{3}} & (k \geqslant k'_L),\\ \alpha\epsilon^{\frac{2}{3}}(k'_L)^{-m-\frac{5}{3}}k^m & (k \leqslant k'_L);\end{cases} \end{aligned}\right\} \quad \text{(C 1)}$$

where it is seen that E_{xx} and E_{zz} have different peak wavenumbers, denoted by k_L and k'_L, respectively. Note, too, that for the sake of simplification we have taken $E_{yy} = E_{zz}$. This simplification is approximately valid for weak shears but not for strong shears. Nevertheless, this model is useful because at this time we are interested in demonstrating how the single parameter $(\langle u_y^2\rangle + \langle u_z^2\rangle)/\langle u_x^2\rangle$ influences β_{ij} and A^N_{ii} rather than in dealing with the complexity of two parameters $\langle u_y^2\rangle/\langle u_x^2\rangle$ and $\langle u_z^2\rangle/\langle u_x^2\rangle$. It would not be too difficult to re-calculate the results of this section for a model in which $E_{yy} \neq E_{zz}$, if that should become desirable. The r.m.s. velocity v_0 of the present model, which we will need later on, is given by

$$\tfrac{2}{3}e_0 = v_0^2 = \tfrac{1}{3}\alpha\epsilon^{\frac{2}{3}}k_L^{-\frac{2}{3}}[1 + 2(k_L/k'_L)^{\frac{2}{3}}][1 + \tfrac{2}{3}(m+1)^{-1}]. \quad \text{(C 2)}$$

With the model spectrum specified by (C 1) we can now evaluate the numerical coefficients β_{ij} defined by (24). This evaluation requires that we do the $\mathbf{k}_1$ and $\mathbf{k}_2$ integrations of (24), and these integrations can be divided into two parts: (1) an integration over spherical angles, and (2) an integration over wavenumbers. However, the angular integrations are not quite as simple as in appendix A because $k'_L \neq k_L$, and, consequently, we cannot take $k_1 \simeq k_2$ everywhere in (24) as we did in several equations of appendix A. Particularly inadequate, for the present case of $k'_L \neq k_L$, is the approximation $k_1/k_2 = 1$ in going from (A 19) to (A 19′), and the approximation $k_2^2(k_1^2+k_2^2)^{-1} = \frac{1}{2}$ in going from (A 18) to (A 18′), because k_1/k_2 takes on the value k_L/k'_L for a significant part of the integration. We do not use those approximations in (24). On the other hand, the use of approximation (A 6) can be justified as adequate for (24). To see why, let us compare approximation (A 6) with the unapproximated form (A 5). In the worst case of $k'_L/k_L = \infty$, we have $k_2/k_1 = \infty$ for a significant part of the integration. In that case the right-hand side of (A 5) is $k_{2z}^2(1-\cos^2\theta)$, whereas the right-hand side of (A 6) is $k_{2z}^2(1-\cos\theta)$. Averaged over θ it can be seen that (A 6) is about 30 % greater than (A 5). Furthermore, $k_1 \simeq k_2$ for an important part of the integration in (A 4) because the factor in (A 5) itself is a maximum. For that part of the integration there is almost zero error. Hence, when integrated over all k_1 and k_2, the approximation (A 6) tends to introduce a positive error of only about $\frac{1}{2}(0+30) = 15$ % for the worst case of $k'_L/k_L = \infty$ (or zero). However, we are only interested in less-extreme cases for which k'_L/k_L does not exceed 2 or 3. In those cases use of the approximation (A 6) results in a positive error of less than 5 or 10 %. Such an error or uncertainty is adequate for the present purposes, considering the uncertainty of the cumulant neglect, and we shall use (A 6) in (24).

With the use of (A 6), the coefficient β_{zz} in (24) is calculated in the same way as in appendix A, except that we do not approximate (A 18) and (A 19) by (A 18′) and (A 19′). Instead, β_{zz} is given by (24) with $\beta_{zz}^{(1)}$, $\beta_{zz}^{(2)}$, $\beta_{zz}^{(3)}$ and $\beta_{zz}^{(4)}$ given respectively by (A 10), (A 13), (A 18) and (A 19):

$$\beta_{zz} = (\tfrac{1}{2}\pi)^{\frac{1}{2}} \frac{v_0 e_0}{\epsilon} \int_0^\infty dk_1 \int_0^\infty dk_2 \frac{k_1^2 k_2^2 E(k_2)\, E_{zz}(k_1)}{(k_1^2+k_2^2)^{\frac{3}{2}}\, e_0 \langle u_z^2\rangle} \times \left[2 - \frac{2\cdot 4 k_2^2}{k_1^2+k_2^2} - 0\cdot 08\left(\frac{4k_1k_2}{k_1^2+k_2^2} - 1\right) + \frac{2}{3}\left(1 - \frac{2k_1^2}{k_1^2+k_2^2}\right)\right]. \quad \text{(C 3)}$$

It is straightforward to evaluate (C 3) with E_{zz} and $E \equiv \frac{1}{2}(E_{xx}+E_{yy}+E_{zz})$ given by (C 1). This can be done easily by computer, but would mask the dependence of β_{zz} on k'_L/k_L. For this reason, we will use a simplifying approximation for the integrations in (C 2) in order to make the dependence of β_{zz} on k'_L/k_L explicit. This approximation is to take $E_{xx}(k) = \langle u_x^2\rangle\,\delta(k-k_L)$, $E_{zz} = \langle u_y^2\rangle\,\delta(k-k'_L)$, $E_{yy} = E_{zz}$ for those integrations. This simplification makes β_{zz} (and all the β_{ij}) too small, because it ignores the large-k contribution, but it gives the ratio of the β_{ij} accurately. In fact, we have also calculated (C 2) without use of the simplification and found that the value of β_{ij} is about twice the value in the simplified case for all i and j. Hence, we use the simplified integration and multiply by a factor of 2. Substitution of the simplified E_{ii} in the integrand of (C 3) and putting $y \equiv k'_L/k_L$ we obtain

$$\beta_{zz} = C^0(1+2y^{-\frac{2}{3}})^{\frac{1}{2}} \left\{0\cdot 72^{-\frac{1}{2}} y^{\frac{1}{3}} + \frac{y^2}{(1+y^2)^{\frac{3}{2}}} \times \left[2 - \frac{2\cdot 4}{1+y^2} - 0\cdot 08\left(\frac{4y}{1+y^2} - 1\right) + \frac{2}{3}\left(1 - \frac{2y^2}{1+y^2}\right)\right]\right\}, \quad \text{(C 4)}$$

where use has been made of (C 2) and the relation $e_0 = \frac{1}{2}\langle u_x^2 \rangle (1+2y^{-\frac{2}{3}})$. (The term $(0{\cdot}72)\, 2^{\frac{1}{2}} y^{\frac{1}{3}}$ in (C 4) comes from the terms $\frac{1}{2}E_{zz}(k_2)\, E_{zz}(k_1) + \frac{1}{2}E_{yy}(k_2)\, E_{zz}(k_1)$ which occur in $E(k_2)\, E_{zz}(k_1)$ of (C 3), whereas the square-bracket term in (C 4) comes from the term $\frac{1}{2}E_{xx}(k_2)\, E_{zz}(k_1)$ of (C 3).) The numerical constant C^0 equals 0·91 for the simplified model and equals 2 for the unsimplified model given by (C 1). It is seen in (C 4) that $y \equiv k'_L/k_L$ has a substantial influence on β_{zz}.

The other coefficients β_{yy}, β_{xx}, β_{xy}, β_{xz} etc. are calculated similarly to β_{zz}. Actually, it is only necessary to calculate β_{zx}, β_{zy} and β_{xx} because all of the remaining β_{ij} can then be determined from symmetry and incompressibility considerations. For β_{zx}, the angular integrations of (24) are given by (A 22), with $\beta_{zx}^{(1)}$, $\beta_{zx}^{(2)}$, $\beta_{zx}^{(3)}$ given respectively by (A 23), (A 26) and (A 27) (the approximations (A 26′) and (A 27′) cannot be used for model (C 1)):

$$\beta_{zx} = -(\tfrac{1}{2}\pi)^{\frac{1}{2}} \frac{v_0 e_0}{\epsilon} \int_0^\infty dk_1 \int_0^\infty dk_2 \frac{k_1^2 k_2^2 E(k_2)\, E_{xx}(k_1)}{(k_1^2+k_2^2)^{\frac{3}{2}}\, e_0 \langle u_x^2 \rangle} \times \left[\frac{(\frac{4}{5})k_2^2}{(k_1^2+k_2^2)} - 0{\cdot}04\left(\frac{4k_1 k_2}{k_1^2+k_2^2} - 1\right) - \frac{2}{3}\left(1 - \frac{2k_1^2}{k_1^2+k_2^2}\right)\right]. \quad \text{(C 5)}$$

Substitution of the delta-function model for E_{ii} into the integrand of (C 5) yields

$$\beta_{zx} = -\tfrac{1}{2} C^0 (1+2y^{-\frac{2}{3}})^{\frac{1}{2}} \left\{(0{\cdot}36)\, 2^{-\frac{3}{2}} + \frac{2y^{\frac{4}{3}}}{(1+y^2)^{\frac{3}{2}}} \times \left[\frac{(\frac{4}{5})\, y^2}{1+y^2} - 0{\cdot}04\left(\frac{4y}{1+y^2} - 1\right) - \frac{2}{3}\left(1 - \frac{2}{1+y^2}\right)\right]\right\}. \quad \text{(C 6)}$$

It can be seen that y has a particularly strong influence on β_{zx}, and can cause the latter to decrease by a factor of 2 when y increases from 1 to 3.

The coefficient β_{zy} is determined from (C 5) by replacing $E_{xx}(k_1)$ with $E_{yy}(k_1)$, and then substituting the delta-function model for E_{ii}. The result is

$$\beta_{zy} = -\tfrac{1}{2} C^0 (1+2y^{-\frac{2}{3}})^{\frac{1}{2}} \left\{(0{\cdot}36)\, 2^{-\frac{1}{2}} y^{\frac{1}{3}} + \left[\frac{4}{5(1+y^2)} - 0{\cdot}04\left(\frac{4y}{1+y^2} - 1\right) - \frac{2}{3}\left(1 - \frac{2y^2}{1+y^2}\right)\right]\right\}, \quad \text{(C 7)}$$

from which it can be seen that β_{zy} increases by only 10 % as y increases from 1 to 2.

Finally, the coefficient β_{xx} is determined from (C 3) by replacing $E_{zz}(k_1)$ with $E_{xx}(k_1)$, and then substituting the delta-function E_{ii} model into the integrand. The result is

$$\beta_{xx} = C^0 (1+2y^{-\frac{2}{3}})^{\frac{1}{2}} \left\{(0{\cdot}72)\, 2^{-\frac{3}{2}} + \frac{2y^{\frac{4}{3}}}{(1+y^2)^{\frac{3}{2}}} \left[2 - \frac{(\frac{12}{5})\, y^2}{1+y^2} - 0{\cdot}08\left(\frac{4y}{1+y^2} - 1\right) + \frac{2}{3}\left(1 - \frac{2}{1+y^2}\right)\right]\right\}, \quad \text{(C 8)}$$

where it can be seen that y has a *very* great influence on β_{xx}, causing the latter to decrease by a factor of 2·3 when y increases from 1 to only 2.

The remaining β_{ij} are determined by symmetry because $E_{yy} = E_{zz}$ in our model:

$$\beta_{yx} = \beta_{zx}, \quad \beta_{yy} = \beta_{zz}, \quad \beta_{yz} = \beta_{zy}, \quad \beta_{xy} = \beta_{xz} = \beta_{zy}. \tag{C 9}$$

Thus, the β_{ij} are all evaluated by (C 4), (C 6)–(C 9) as functions of $y \equiv k'_L/k_L$. The simplified model gives $C^0 = 0{\cdot}91$; the unsimplified model gives approximately the same values for β_{ij} with $C^0 = 2$. These expressions for β_{ij} show that variations of y cause large variations of β_{xx}, β_{zx} and β_{yx}, moderate variations of β_{zz} and β_{yy}, and small variations of β_{xy}, β_{xz}, β_{yz} and β_{zy}. It can also be shown that the calculated β_{ij} satisfy

$$\sum_{i,\,y}^{x,\,y,\,z} \beta_{ij} = 0$$

which is a consequence of the incompressibility condition

$$\sum_{i}^{x,\,y,\,z} A^N_{ii} = 0.$$

REFERENCES

Champagne, F. H., Harris, V. G. & Corrsin, S. 1970 *J. Fluid Mech.* **41**, 81–139.

Comte-Bellot, G. & Corrsin, S. 1966 *J. Fluid Mech.* **25**, 657–682.

Comte-Bellot, G. & Corrsin, S. 1971 *J. Fluid Mech.* **48**, 273–337.

Hanjelic, K. & Launder, B. E. 1972 *J. Fluid Mech.* **52**, 609–638.

Harris, V. G., Graham, J. A. H. & Corrsin, S. 1977 *J. Fluid Mech.* **81**, 657–687.

Herring, J. R. 1974 *Phys. Fluids* **17**, 589–872.

Kaimal, J. C., Wyngaard, J. C., Izumi, Y. & Cote, O. R. 1972 *Quart. J. Roy. Met. Soc.* **98**, 563–589.

Kraichman, R. 1959 *J. Fluid Mech.* **5**, 497–543.

Launder, B. E., Reece, G. J. & Rodi, W. 1975 *J. Fluid Mech.* **52**, 537–566.

Leslie, D. C. 1973 *Developments in the Theory of Turbulence*. Clarendon.

Lumley, J. L. & Newman, G. R. 1977 *J. Fluid Mech.* **82**, 161–178.

Lumley, J. L. & Khajeh-Nouri, B. 1974 *Adv. Geophys.* **18**A, 169–192.

Proudman, I. & Reid, W. H. 1954 *Phil. Trans. R. Soc. Lond* A **247**, 163.

Reynolds, W. C. 1976 *Ann. Rev. Fluid Mech.* **8**, 183–208.

Rotta, J. 1951 *Z. Phys.* **129**, 547–572.

Schumann, V. & Herring, J. R. 1976 *J. Fluid Mech.* **76**, 755–782.

Weinstock, J. 1981 *J. Fluid Mech.* **105**, 369–395.

Wyngaard, J. C. 1980 In *Turbulent Shear Flows* 2 (ed. J. S. Bradley, F. Durst, B. E. Launder, F. W. Schmidt & J. H. Whitelaw). Springer.